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Extended IOWG Operator and its Use in Group Decision Making Based on Multiplicative Linguistic Preference Relations

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Abstract: In [1], Xu and Da introduced the Induced Ordered Weighted Geometric (IOWG) operator, which takes as its argument pairs, called OWG pairs, in which one component is used to induce an ordering over the second components which are exact numerical values and then aggregated. In this study, we develop an extended IOWG (EIOWG) operator, in which the second components are linguistic variables. We study some desirable properties of the EIOWG operator, and then apply the EIOWG operator to group decision making based on multiplicative linguistic preference relations.

Keywords: Aggregation, Induced Ordered Weighted Geometric (IOWG) Operator, Group Decision Making, Multiplicative Linguistic Preference Relation

INTRODUCTION

The ordered weighted averaging (OWA) operator was developed by Yager [2]. The fundamental aspect of the OWA operator is a reordering step in which the input arguments are rearranged in descending order [2-8]. The ordered weighted geometric (OWG) operator is an aggregation operator that is based on the OWA operator and the geometric mean [1,9-15]. Yager and Filev [16] introduced a more general type of OWA operator called induced ordered weighted averaging (IOWA) operator. The IOWA operator takes as its argument pairs, called OWA pairs, in which one component is used to induce an ordering over the second components which are exact numerical values and then aggregated. Recently, Xu and Da [1] developed an induced ordered weighted geometric (IOWG) operator that is based on the IOWA operator and the geometric mean, which can be used to aggregate multiplicative preference relations with exact numerical values in group decision making problems [17]. However, in many situations, the input arguments take the form of linguistic variables rather than numerical ones [13,18-38]. Therefore, it is necessary to pay attention to this issue. In this study, we shall develop an extended IOWG (EIOWG) operator, and study some desirable properties of the EIOWG operator. Then, we shall develop an approach, based on the EIOWG and the extended OWG (EOWG) operators, for ranking alternatives in group decision making with multiplicative linguistic preference relations. Finally, we shall apply the developed approach to the evaluation of investment alternatives of an investment company and draw our conclusions.

EIOWG Operator: Let $S = \{ s_\alpha \mid \alpha = 1/t, ..., 1/2, 1, 2, ..., t\}$ be a multiplicative linguistic term set with odd cardinality. Any label, s_α, represents a possible value for a linguistic variable, and it should satisfy the following characteristics:

1. The set is ordered: $s_\alpha > s_\beta$ if $\alpha > \beta$;
2. There is the reciprocal operator: rec $(s_\alpha) = s_\beta$ such that $\alpha\beta = 1$.

We call this multiplicative linguistic term set S the multiplicative linguistic scale. For example, S can be defined as:

$$S = \{ s_{1/5} = extremely\ low,\ s_{1/4} = very\ low,\ s_{1/3} = low,\ s_{1/2} = slightly\ low,\ s_1 = medium,\ s_2 = slightly\ high,\ s_3 = high,\ s_4 = very\ high,\ s_5 = extremely\ high \}$$

To preserve all the given information, we extend the discrete multiplicative linguistic term set S to a continuous multiplicative linguistic term set $\bar{S} = \{ s_\alpha \mid \alpha \in [1/q, q]\}$, where $q\ (q > t)$ is a sufficiently large positive integer. If $s_\alpha \in S$, then we call s_α the original linguistic term, otherwise, we call s_α the virtual linguistic term. In general, the decision maker uses the original linguistic terms to evaluate alternatives, and the virtual linguistic terms can only appear in calculation.

Let $s_\alpha, s_\beta \in \bar{S}$, and $\mu, \mu_1, \mu_2 \in [0,1]$, we give some operational laws as follows [37]:

1. $(s_\alpha)^\mu = s_{\alpha^\mu}$;

2. $(s_\alpha)^{\mu_1} \otimes (s_\alpha)^{\mu_2} = (s_\alpha)^{\mu_1+\mu_2}$;

3. $(s_\alpha \otimes s_\beta)^\mu = (s_\alpha)^\mu \otimes (s_\beta)^\mu$;

4. $s_\alpha \otimes s_\beta = s_\beta \otimes s_\alpha = s_{\alpha\beta}$.

The ordered weighted geometric (OWG) operator is an aggregation operator that Chiclana *et al*. [9] defined and characterized to design multiplicative decision-making models [10,11,14]. It is based on the ordered weighted averaging (OWA) operator [2] and on the geometric mean. Xu and Da [12] presented some families of OWG operators.

Definition 1[1,9-15]: An OWG operator of dimension n is a mapping $OWG : R^{+^n} \to R^+$ which has associated with it an exponential weighting vector $w = (w_1, w_2, ..., w_n)^T$, with $w_j \in [0,1]$ and

$\sum_{j=1}^{n} w_j = 1$, such that

$$OWG_w(\alpha_1, \alpha_2 ..., \alpha_n) = \prod_{j=1}^{n} b_j^{w_j} \tag{1}$$

where b_j is the j th largest of the α_i, R^+ is the positive real number set.

The OWG operator has only been used in situations in which the input arguments are the exact numerical values. However, judgements of people depend on personal psychological aspects such as experience, learning, situation, state of mind, and so forth. It is more suitable to provide their preferences by means of linguistic variables rather than numerical ones (for example when evaluating the comfort or design of a car, terms like good, fair, poor can be used). In [37], Xu extended the OWG operator to accommodate the situations where the input arguments are linguistic variables.

Definition 2 [37]: An extended ordered weighted geometric (EOWG) operator of dimension n is a mapping $EOWG : \bar{S}^n \to \bar{S}$, which has associated with it an exponential weighting vector $w = (w_1, w_2, ..., w_n)^T$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that;

$$\begin{aligned} EOWG_w\ (s_{\alpha_1}, s_{\alpha_2}, ..., s_{\alpha_n}) &= (s_{\beta_1})^{w_1} \otimes (s_{\beta_2})^{w_2} \otimes \cdots \otimes (s_{\beta_n})^{w_n} \\ &= (s_{\beta_1^{w_1}}) \otimes (s_{\beta_2^{w_2}}) \otimes \cdots \otimes (s_{\beta_n^{w_n}}) \\ &= s_{\bar{\beta}} \end{aligned}$$

where $\beta = \prod_{j=1}^{n} \beta_j^{w_j}$, s_{β_j} is the j th largest of the s_{α_j}, which is an extension of the OWG operator.

Example 1: Assume $w = (0.3, 0.2, 0.4, 0.1)^T$, then

$$\begin{aligned} EOWG_w\ (s_{1/2}, s_4, s_{1/3}, s_5) &= (s_5)^{0.3} \otimes (s_4)^{0.2} \otimes (s_{1/2})^{0.4} \otimes (s_{1/3})^{0.1} \\ &= (s_{5^{0.3}}) \otimes (s_{4^{0.1}}) \otimes (s_{(1/2)^{0.4}}) \otimes (s_{(1/3)^{0.2}}) \\ &= s_{1.45} \end{aligned}$$

In [2], Yager defined the concept of the ordered weighted averaging (OWA) operator. Later, Yager and Filev [16] introduced a more general type of OWA operator called induced ordered weighted averaging (IOWA) operator. The

IOWA operator takes as its argument pairs, called OWA pairs, in which one component is used to induce an ordering over the second components which are exact numerical values and then aggregated. Xu and Da [1] developed an induced ordered weighted geometric (IOWG) operator that is based on the IOWA operator and the geometric mean, which can be used to aggregate multiplicative preference relations with exact numerical values in group decision-making problems.

Definition 3 [1]: An IOWG operator is defined as follows:

$$IOWG_w(\langle u_1,a_1\rangle,\langle u_2,a_2\rangle ...,\langle u_n,a_n\rangle)=\prod_{j=1}^{n} b_j^{w_j} \tag{2}$$

where $w=(w_1,w_2,...,w_n)^T$ is an exponential weighting vector, such that $w_j\in[0,1]$, $\sum_{j=1}^{n} w_j=1$, b_j is the a_i value of the OWG pair $\langle u_i,a_i\rangle$ having the j th largest u_i, and u_i in $\langle u_i,a_i\rangle$ is referred to as the order inducing variable and a_i as the argument variable, $a_i\in R^+$, $i=1,2,...,n$, R^+ is the real set. Especially, if $w=(1/n,1/n,...,1/n)^T$, then $IOWG$ is reduced to the geometric mean operator; if $u_i=a_i$, for all i, then $IOWG$ is reduced to the OWG operator; if $u_i=No.\,i$, for all i, where $No.\,i$ is the ordered position of the a_i, then $IOWG$ is the weighted geometric mean operator.

In the following, we shall extend the IOWG operator to accommodate the situations where the input arguments are linguistic variables.

Definition 4: An extended IOWG (EIOWG) operator is defined as follows:

$$EIOWG_w(\langle u_1,s_{\alpha_1}\rangle,\langle u_2,s_{\alpha_2}\rangle ...,\langle u_n,s_{\alpha_n}\rangle)=(s_{\gamma_1})^{w_1}\otimes(s_{\gamma_2})^{w_2}\otimes\cdots\otimes(s_{\gamma_n})^{w_n}=s_{\bar{\gamma}} \tag{3}$$

where $\bar{\gamma}=\prod_{j=1}^{n}\gamma_j^{w_j}$, $w=(w_1,w_2,...,w_n)^T$ is an exponential weighting vector, such that $w_j\in[0,1]$, $\sum_{j=1}^{n} w_j=1$, s_{γ_j} is the s_{α_i} value of the OWG pair $\langle u_i,s_{\alpha_i}\rangle$ having the j th largest u_i, and u_i in $\langle u_i,s_{\alpha_i}\rangle$ is referred to as the order inducing variable and s_{α_i} as the multiplicative linguistic argument variable. Especially, if $w=(1/n,1/n,...,1/n)^T$, then $EIOWG$ is reduced to the extended geometric mean operator; if $u_i=s_{\alpha_i}$, for all i, then $EIOWG$ is reduced to the EOWG operator; if $u_i=No.\,i$, for all i, where $No.\,i$ is the ordered position of the s_i, then $EIOWG$ is reduced to the extended weighted geometric mean operator.

Example 2: Consider a collection of four OWG pairs $\langle No.1,s_{1/3}\rangle$, $\langle No.3,s_4\rangle$, $\langle No.4,s_{1/2}\rangle$, and $\langle No.2,s_5\rangle$, we desire to aggregate using the weighting vector $w=(0.2,0.3,0.3,0.2)^T$. Performing the ordering the OWG pairs with respect to the first component, we get

$\langle No.1,s_{1/3}\rangle$, $\langle No.2,s_5\rangle$, $\langle No.3,s_4\rangle$, $\langle No.4,s_{1/2}\rangle$

This ordering induces the ordered linguistic arguments

$s_{\gamma_1}=s_{1/3}$, $s_{\gamma_2}=s_5$, $s_{\gamma_3}=s_4$, $s_{\gamma_4}=s_{1/2}$

and from this, we get an aggregated value

$$EIOWG_w(\langle No.1,s_{1/3}\rangle,\langle No.3,s_4\rangle,\langle No.4,s_{1/2}\rangle,\langle No.2,s_5\rangle)=(s_{1/3})^{0.2}\otimes(s_5)^{0.3}\otimes(s_4)^{0.3}\otimes(s_{1/2})^{0.2}$$

$$= s_{(1/3)^{0.2}} \otimes s_{5^{0.3}} \otimes s_{4^{0.3}} \otimes s_{(1/2)^{0.2}} = s_{1.72}$$

Example 3: Consider the following collection of OWG pairs $\langle 0.3, s_{1/4}\rangle$, $\langle 0.1, s_5\rangle$, $\langle 0.4, s_{1/2}\rangle$, $\langle 0.6, s_3\rangle$
Performing the ordering the OWG pairs with respect to the first component, we have $\langle 0.6, s_3\rangle$, $\langle 0.4, s_{1/2}\rangle$, $\langle 0.3, s_{1/4}\rangle$, $\langle 0.1, s_5\rangle$

This ordering induces the ordered linguistic arguments $s_{\gamma_1} = s_3$, $s_{\gamma_2} = s_{1/2}$, $s_{\gamma_3} = s_{1/4}$, $s_{\gamma_4} = s_5$
If the weighting vector $w = (0.1, 0.4, 0.3, 0.2)^T$, then we get an aggregated value

$$EIOWG_w(\langle 0.3, s_{1/4}\rangle, \langle 0.1, s_5\rangle, \langle 0.4, s_{1/2}\rangle, \langle 0.6, s_3\rangle) = (s_3)^{0.1} \otimes (s_{1/2})^{0.4} \otimes (s_{1/4})^{0.3} \otimes (s_5)^{0.2}$$

$$= s_{3^{0.1}} \otimes s_{(1/2)^{0.4}} \otimes s_{(1/4)^{0.3}} \otimes s_{5^{0.2}} = s_{0.77}$$

However, if we replace the objects in Example 3 with $\langle 0.1, s_{1/4}\rangle$, $\langle 0.1, s_5\rangle$, $\langle 0.4, s_{1/2}\rangle$, $\langle 0.6, s_3\rangle$

then there is a tie between $\langle 0.1, s_{1/4}\rangle$ and $\langle 0.1, s_5\rangle$ with respect to order inducing variable. In this case, we can follow the policy presented by Yager and Filev [16], that is, to replace the arguments of the tied objects by the average of the arguments of the tied objects. Thus, for Example 3, we replace the argument component of each of $\langle 0.1, s_{1/4}\rangle$ and $\langle 0.1, s_5\rangle$ by their geometric mean $(s_{1/4} \otimes s_5)^{1/2} = s_{1.12}$. This substitution gives us ordered linguistic arguments

$s_{\gamma_1} = s_3$, $s_{\gamma_2} = s_{1/2}$, $s_{\gamma_3} = s_{1.12}$, $s_{\gamma_4} = s_{1.12}$

thus $\mathrm{EIOWG}_w(\langle 0.1, s_{1/4}\rangle, \langle 0.1, s_5\rangle, \langle 0.4, s_{1/2}\rangle, \langle 0.6, s_3\rangle) = (s_3)^{0.1} \otimes (s_{1/2})^{0.4} \otimes (s_{1.12})^{0.3} \otimes (s_{1.12})^{0.2}$

$$= s_{3^{0.1}} \otimes s_{(1/2)^{0.4}} \otimes s_{1.12^{0.3}} \otimes s_{1.12^{0.3}} = s_{0.91}$$

If k items are tied, we replace these by k replicas of their geometric mean.
In the following, we shall study some desirable properties of the EIOWG operator.

Theorem 1 (Commutativity):

$$EIOWG_w(\langle u_1, s_{\alpha_1}\rangle, \langle u_2, s_{\alpha_2}\rangle, \ldots, \langle u_n, s_{\alpha_n}\rangle) = EIOWG_w(\langle u_1', s_{\alpha_1}'\rangle, \langle u_2', s_{\alpha_2}'\rangle, \ldots, \langle u_n', s_{\alpha_n}'\rangle)$$

$$EIOWG_w(\langle u_1, s_{\alpha_1}\rangle, \langle u_2, s_{\alpha_2}\rangle, \ldots, \langle u_n, s_{\alpha_n}\rangle) = EIOWG_w(\langle u_1', s_{\alpha_1}'\rangle, \langle u_2', s_{\alpha_2}'\rangle, \ldots, \langle u_n', s_{\alpha_n}'\rangle)$$

where $(\langle u_1', s_{\alpha_1}'\rangle, \langle u_2', s_{\alpha_2}'\rangle, \ldots, \langle u_n', s_{\alpha_n}'\rangle)$ is any permutation of $(\langle u_1, s_{\alpha_1}\rangle, \langle u_2, s_{\alpha_2}\rangle, \ldots, \langle u_n, s_{\alpha_n}\rangle)$.

Proof. Let

$$EIOWG_w(\langle u_1, s_{\alpha_1}\rangle, \langle u_2, s_{\alpha_2}\rangle, \ldots, \langle u_n, s_{\alpha_n}\rangle) = (s_{\gamma_1})^{w_1} \otimes (s_{\gamma_2})^{w_2} \otimes \cdots \otimes (s_{\gamma_n})^{w_n}$$

$$EIOWG_w\left(\langle u_1^{'}, s_{\alpha_1}^{'}\rangle, \langle u_2^{'}, s_{\alpha_2}^{'}\rangle, \ldots, \langle u_n^{'}, s_{\alpha_n}^{'}\rangle\right) = (s_{\gamma_1}^{'})^{w_1} \otimes (s_{\gamma_2}^{'})^{w_2} \otimes \cdots \otimes (s_{\gamma_n}^{'})^{w_n}$$

Since $\left(\langle u_1^{'}, s_{\alpha_1}^{'}\rangle, \langle u_2^{'}, s_{\alpha_2}^{'}\rangle, \ldots, \langle u_n^{'}, s_{\alpha_n}^{'}\rangle\right)$ is a permutation of $\left(\langle u_1, s_{\alpha_1}\rangle, \langle u_2, s_{\alpha_2}\rangle, \ldots, \langle u_n, s_{\alpha_n}\rangle\right)$, we have $s_{\gamma_j} = s_{\gamma_j}^{'}$ $(j = 1,2,\ldots,n)$, then

$$EIOWG_w\left(\langle u_1, s_{\alpha_1}\rangle, \langle u_2, s_{\alpha_2}\rangle, \ldots, \langle u_n, s_{\alpha_n}\rangle\right) = EIOWG_w\left(\langle u_1^{'}, s_{\alpha_1}^{'}\rangle, \langle u_2^{'}, s_{\alpha_2}^{'}\rangle, \ldots, \langle u_n^{'}, s_{\alpha_n}^{'}\rangle\right)$$

Theorem 2 (Idempotency): If $s_{\alpha_j} = s_\alpha$, for all j, then

$$EIOWG_w\left(\langle u_1, s_{\alpha_1}\rangle, \langle u_2, s_{\alpha_2}\rangle, \ldots, \langle u_n, s_{\alpha_n}\rangle\right) = s_\alpha$$

Proof: Since $s_{\alpha_j} = s_\alpha$, for all j, we have

$$EIOWG_w\left(\langle u_1, s_{\alpha_1}\rangle, \langle u_2, s_{\alpha_2}\rangle, \ldots, \langle u_n, s_{\alpha_n}\rangle\right) = (s_{\gamma_1})^{w_1} \otimes (s_{\gamma_2})^{w_2} \otimes \cdots \otimes (s_{\gamma_n})^{w_n}$$

$$= (s_\alpha)^{w_1} \otimes (s_\alpha)^{w_2} \otimes \cdots \otimes (s_\alpha)^{w_n}$$

$$= (s_\alpha)^{\sum_{j=1}^{n} w_j} = s_\alpha$$

Theorem 3 (Monotonicity): If $s_{\alpha_j} \le \hat{s}_{\alpha_j}$, for all j, then

$$EIOWG_w\left(\langle u_1, s_{\alpha_1}\rangle, \langle u_2, s_{\alpha_2}\rangle, \ldots, \langle u_n, s_{\alpha_n}\rangle\right) \le EIOWG_w\left(\langle u_1, \hat{s}_{\alpha_1}\rangle, \langle u_2, \hat{s}_{\alpha_2}\rangle, \ldots, \langle u_n, \hat{s}_{\alpha_n}\rangle\right)$$

Proof: Let

$$EIOWG_w\left(\langle u_1, s_{\alpha_1}\rangle, \langle u_2, s_{\alpha_2}\rangle, \ldots, \langle u_n, s_{\alpha_n}\rangle\right) = (s_{\gamma_1})^{w_1} \otimes (s_{\gamma_2})^{w_2} \otimes \cdots \otimes (s_{\gamma_n})^{w_n}$$

$$EIOWG_w\left(\langle u_1, \hat{s}_{\alpha_1}\rangle, \langle u_2, \hat{s}_{\alpha_2}\rangle, \ldots, \langle u_n, \hat{s}_{\alpha_n}\rangle\right) = (\hat{s}_{\gamma_1})^{w_1} \otimes (\hat{s}_{\gamma_2})^{w_2} \otimes \cdots \otimes (\hat{s}_{\gamma_n})^{w_n}$$

Since $s_{\alpha_j} \le \hat{s}_{\alpha_j}$, for all j, it follows that $s_{\gamma_j} \le \hat{s}_{\gamma_j}$, then

$$EIOWG_w\left(\langle u_1, s_{\alpha_1}\rangle, \langle u_2, s_{\alpha_2}\rangle, \ldots, \langle u_n, s_{\alpha_n}\rangle\right) \le EIOWG_w\left(\langle u_1, \hat{s}_{\alpha_1}\rangle, \langle u_2, \hat{s}_{\alpha_2}\rangle, \ldots, \langle u_n, \hat{s}_{\alpha_n}\rangle\right)$$

Theorem 4 (Bounded): $\underset{j}{Min}(s_{\alpha_j}) \le EIOWG_w\left(\langle u_1, s_{\alpha_1}\rangle, \langle u_2, s_{\alpha_2}\rangle, \ldots, \langle u_n, s_{\alpha_n}\rangle\right) \le \underset{j}{Max}(s_{\alpha_j})$

Proof: Let $\underset{j}{Max}(s_{\alpha_j}) = s_\beta$ and $\underset{j}{Min}(s_{\alpha_j}) = s_\alpha$, then

$$EIOWG_w\left(\langle u_1, s_{\alpha_1}\rangle, \langle u_2, s_{\alpha_2}\rangle, \ldots, \langle u_n, s_{\alpha_n}\rangle\right) = (s_{\gamma_1})^{w_1} \otimes (s_{\gamma_2})^{w_2} \otimes \cdots \otimes (s_{\gamma_n})^{w_n}$$

$$\le (s_\beta)^{w_1} \otimes (s_\beta)^{w_2} \otimes \cdots \otimes (s_\beta)^{w_n} = (s_\beta)^{\sum_{j=1}^{n} w_j} = s_\beta$$

$$EIOWG_w\left(\langle u_1, s_{\alpha_1}\rangle, \langle u_2, s_{\alpha_2}\rangle, ..., \langle u_n, s_{\alpha_n}\rangle\right) = (s_{\gamma_1})^{w_1} \otimes (s_{\gamma_2})^{w_2} \otimes \cdots \otimes (s_{\gamma_n})^{w_n}$$

$$\geq (s_\alpha)^{w_1} \otimes (s_\alpha)^{w_2} \otimes \cdots \otimes (s_\alpha)^{w_n} = (s_\alpha)^{\sum_{j=1}^n w_j} = s_\alpha$$

hence $\underset{j}{Min}(s_{\alpha_j}) \leq EIOWG_w\left(\langle u_1, s_{\alpha_1}\rangle, \langle u_2, s_{\alpha_2}\rangle, ..., \langle u_n, s_{\alpha_n}\rangle\right) \leq \underset{j}{Max}(s_{\alpha_j})$

An Approach Based on the EIOWG and the EOWG Operators to Group Decision Making with Multiplicative Linguistic Preference Relations: Consider a group decision making problem with linguistic preference information. Let $X = \{x_1, x_2, ..., x_n\}$ be the set of alternatives, and $D = \{d_1, d_2, ..., d_m\}$ be the set of decision makers. Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)^T$ be the weight vector of decision makers, where $\lambda_l \geq 0, \sum_{l=1}^m \lambda_l = 1$. The decision maker $d_l \in D$ compares these alternatives with respect to a single criterion by the multiplicative linguistic terms in the set $S = \{s_\alpha \mid \alpha = 1/t, ..., 1/2, 1, 2, ..., t\}$, and constructs the multiplicative linguistic preference relation $R^{(l)} = (r_{ij}^{(l)})_{n\times n}$, whose element $r_{ij}^{(l)}$ estimates the preference degree of alternative x_i over x_j, and meets

$s_{1/t} \leq r_{ij}^{(l)} \leq s_t$, $r_{ij}^{(l)} \otimes r_{ji}^{(l)} = s_1$, $r_{ii}^{(l)} = s_1$, for all $i, j = 1,2,...,n$

It is well known that the multiplicative preference relations to express the judgements are reciprocal, however, In [14], Herrera and Herrera-Viedma showed that reciprocity generally is not preserved when aggregating multiplicative preference relations using the OWG operator. In the section, we shall show that the reciprocal property can be maintained when aggregating multiplicative linguistic preference relations using the EIOWG operator, where the order inducing variable $u_l = \lambda_l$ ($l = 1,2,...,m$).

Theorem 5: Let $R^{(1)}, R^{(2)}, ..., R^{(m)}$ be multiplicative linguistic preference relations provided by m decision makers $d_l (l = 1,2,...,m)$, where $R^{(l)} = (r_{ij}^{(l)})_{n\times n}$, $r_{ij}^{(l)} \in S$ ($l = 1,2,...,m; i, j = 1,2,...,n$), then their collective linguistic preference relation $\hat{R} = (\hat{r}_{ij})_{n\times n}$ is also a multiplicative linguistic preference relation with

$\hat{r}_{ij} = EIOWG_w\left(\langle \lambda_1, r_{ij}^{(1)}\rangle, \langle \lambda_2, r_{ij}^{(2)}\rangle, ..., \langle \lambda_m, r_{ij}^{(m)}\rangle\right) = (b_{ij}^{(1)})^{w_1} \otimes (b_{ij}^{(2)})^{w_2} \otimes \cdots \otimes (b_{ij}^{(m)})^{w_m}$ where $b_{ij}^{(k)}$ is the $r_{ij}^{(l)}$ value of the OWG pair $\langle \lambda_l, r_{ij}^{(l)}\rangle$ having the k th largest λ_l, $s_{1/t} \leq \hat{r}_{ij} \leq s_t$, $\hat{r}_{ij} \otimes \hat{r}_{ji} = s_1$, $\hat{r}_{ii} = s_1$, for all $i, j = 1,2,...,n$.

Proof: Since $R^{(1)}, R^{(2)}, ..., R^{(m)}$ are multiplicative linguistic preference relations, we have $s_{1/t} \leq r_{ij}^{(l)} \leq s_t$ and $r_{ij}^{(l)} \otimes r_{ji}^{(l)} = s_1$, $r_{ii}^{(l)} = s_1$, for all $l = 1,2,...,m;\ i, j = 1,2,...,n$, and then

$$\hat{r}_{ij} = EIOWG_w\left(\langle \lambda_1, r_{ij}^{(1)}\rangle, \langle \lambda_2, r_{ij}^{(2)}\rangle, ..., \langle \lambda_m, r_{ij}^{(m)}\rangle\right) = (b_{ij}^{(1)})^{w_1} \otimes (b_{ij}^{(2)})^{w_2} \otimes \cdots \otimes (b_{ij}^{(m)})^{w_m}$$

$$\geq (s_{1/t})^{w_1} \otimes (s_{1/t})^{w_2} \otimes \cdots \otimes (s_{1/t})^{w_m} = (s_{1/t})^{\sum_{l=1}^m w_l} = s_{1/t}$$

$$\hat{r}_{ij} = EIOWG_w\left(\langle \lambda_1, r_{ij}^{(1)}\rangle, \langle \lambda_2, r_{ij}^{(2)}\rangle, ..., \langle \lambda_m, r_{ij}^{(m)}\rangle\right) = (b_{ij}^{(1)})^{w_1} \otimes (b_{ij}^{(2)})^{w_2} \otimes \cdots \otimes (b_{ij}^{(m)})^{w_m}$$

$$\leq (s_t)^{w_1} \otimes (s_t)^{w_2} \otimes \cdots \otimes (s_t)^{w_m} = (s_t)^{\sum_{l=1}^{m} w_l} = s_t$$

$$\hat{r}_{ij} \otimes \hat{r}_{ji} = \left((b_{ij}^{(1)})^{w_1} \otimes (b_{ij}^{(2)})^{w_2} \otimes \cdots \otimes (b_{ij}^{(m)})^{w_m}\right) \otimes \left((b_{ji}^{(1)})^{w_1} \otimes (b_{ji}^{(2)})^{w_2} \otimes \cdots \otimes (b_{ji}^{(m)})^{w_m}\right)$$

$$= ((b_{ij}^{(1)} \otimes b_{ji}^{(1)})^{w_1} \otimes (b_{ij}^{(2)} \otimes b_{ji}^{(2)})^{w_2} \otimes \cdots \otimes (b_{ij}^{(m)} \otimes b_{ji}^{(m)}))^{w_m}$$

$$= (s_1)^{w_1} \otimes (s_1)^{w_2} \otimes \cdots \otimes (s_1)^{w_m} = (s_1)^{\sum_{l=1}^{m} w_l} = s_1$$

$$\hat{r}_{ii} = (b_{ii}^{(1)})^{w_1} \otimes (b_{ii}^{(2)})^{w_2} \otimes \cdots \otimes (b_{ii}^{(m)})^{w_m} = (r_{ii}^{(1)})^{w_1} \otimes (r_{ii}^{(2)})^{w_2} \otimes \cdots \otimes (r_{ii}^{(m)})^{w_m}$$

$= (s_1)^{w_1} \otimes (s_1)^{w_2} \otimes \cdots \otimes (s_1)^{w_m} = (s_1)^{\sum_{l=1}^{m} w_l} = s_1$ thus, $\hat{R}$ is a multiplicative linguistic preference relation.

This completes the proof of Theorem 5.

In the following, we shall apply the EIOWG and the EOWG operators to group decision making based on multiplicative linguistic preference relations.

Step 1: For a group decision making problem with linguistic preference information. The decision maker $d_l \in D$ compares these alternatives with respect to a single criterion by the multiplicative linguistic terms in S, and constructs the multiplicative linguistic preference relation $R^{(l)} = (r_{ij}^{(l)})_{n\times n}$, where $s_{1/t} \leq r_{ij}^{(l)} \leq s_t$, $r_{ij}^{(l)} \otimes r_{ji}^{(l)} = s_1$, $r_{ii}^{(l)} = s_1$, for all $i, j = 1,2,...,n$.

Step 2: Utilize the EIOWG operator $\hat{r}_{ij} = EIOWG_w\left(\langle \lambda_1, r_{ij}^{(1)} \rangle, \langle \lambda_2, r_{ij}^{(2)} \rangle, \cdots, \langle \lambda_m, r_{ij}^{(m)} \rangle\right),\ i, j = 1,2,...,n$ to aggregate all the multiplicative linguistic preference relations $R^{(l)} = (r_{ij}^{(l)})_{n\times n}$ $(l = 1,2,...,m)$ to get the collective multiplicative linguistic preference relation $\hat{R} = (\hat{r}_{ij})_{n\times n}$.

Step 3: Utilize the EOWG operator $\hat{r}_i = EOWG_w(\hat{r}_{i1}, \hat{r}_{i2}, \cdots, \hat{r}_{in})$, $i = 1,2,...,n$ to aggregate $\hat{r}_{ij}$ $(j = 1,2,..., n)$ corresponding to the alternative x_i, and then get the collective linguistic preference degree $\hat{r}_i$ $(i = 1,2,..., n)$ of the i th alternative over all the other alternatives.

Step 4: Rank all the alternatives and select the best one(s) in accordance with the values of $\hat{r}_i (i = 1,2,...,n)$.

Step 5: End.

Illustrative Example: Let us suppose an investment company, which wants to invest a sum of money in the best option (adapted from [32]). There is a panel with five possible alternatives in which to invest the money:

1) x_1 is a car industry; 2) x_2 is a food company; 3) x_3 is a computer company; 4) x_4 is an arms company;

5) x_5 is a TV company.

One main criterion used is growth analysis. There are three decision makers $d_l (l = 1,2,3)$, whose weight vector

$\lambda = (0.5, 0.3, 0.2)^T$. The decision makers compare these five companies with respect to the criterion growth analysis by using the multiplicative linguistic scale

$S = \{ s_{1/5} = \text{extremely low},\ s_{1/4} = \text{very low},\ s_{1/3} = \text{low},\ s_{1/2} = \textit{slightly low},\ s_1 = \textit{medium},$

$s_2 = \textit{slightly high},\ s_3 = \text{high},\ s_4 = \text{very high},\ s_5 = \text{extremely high} \}$

and construct, respectively, the multiplicative linguistic preference relations $R^{(l)}$ $(l = 1,2,3)$ as listed in Tables 1-3.

Table 1: Multiplicative Linguistic Preference Relation $R^{(1)}$

	x_1	x_2	x_3	x_4	x_5
x_1	s_1	s_3	$s_{1/4}$	s_3	$s_{1/4}$
x_2	$s_{1/3}$	s_1	s_4	s_2	$s_{1/3}$
x_3	s_4	$s_{1/4}$	s_1	s_3	s_5
x_4	$s_{1/3}$	$s_{1/2}$	$s_{1/3}$	s_1	s_4
x_5	s_4	s_3	$s_{1/5}$	$s_{1/4}$	s_1

Table 2: Multiplicative Linguistic Preference Relation $R^{(2)}$

	x_1	x_2	x_3	x_4	x_5
x_1	s_1	s_4	$s_{1/3}$	s_3	$s_{1/2}$
x_2	$s_{1/4}$	s_1	s_4	s_4	$s_{1/3}$
x_3	s_3	$s_{1/4}$	s_1	s_5	s_4
x_4	$s_{1/3}$	$s_{1/4}$	$s_{1/5}$	s_1	s_5
x_5	s_2	s_3	$s_{1/4}$	$s_{1/5}$	s_1

Table 3: Multiplicative Linguistic Preference Relation $R^{(3)}$

	x_1	x_2	x_3	x_4	x_5
x_1	s_1	s_3	$s_{1/2}$	s_3	$s_{1/5}$
x_2	$s_{1/3}$	s_1	s_5	s_4	$s_{1/4}$
x_3	s_2	$s_{1/5}$	s_1	s_5	s_3
x_4	$s_{1/3}$	$s_{1/4}$	$s_{1/5}$	s_1	s_4
x_5	s_5	s_4	$s_{1/3}$	$s_{1/4}$	s_1

To get the most desirable alternative(s), the following steps are involved:

Step 1: Utilize the EIOWG operator (let its weighting vector be $w = (0.3, 0.4, 0.3)^T$)

$$\hat{r}_{ij} = EIOWG_w\left(\left\langle \lambda_1, r_{ij}^{(1)} \right\rangle, \left\langle \lambda_2, r_{ij}^{(2)} \right\rangle, \left\langle \lambda_3, r_{ij}^{(3)} \right\rangle\right),\ i, j = 1,2,3,4,5$$

to aggregate all the multiplicative linguistic preference relations $R^{(l)} = (r_{ij}^{(l)})_{5\times5}$ $(l = 1,2,3)$

$$\hat{r}_{11} = EIOWG_w\left(\left\langle 0.5, r_{11}^{(1)} \right\rangle, \left\langle 0.3, r_{11}^{(2)} \right\rangle, \left\langle 0.2, r_{11}^{(3)} \right\rangle\right) = (s_1)^{0.3} \otimes (s_1)^{0.4} \otimes (s_1)^{0.3} = s_1$$

Similarly, we have

$\hat{r}_{12} = s_{3.37}$, $\hat{r}_{13} = s_{0.35}$, $\hat{r}_{14} = s_3$, $\hat{r}_{15} = s_{0.31}$

$\hat{r}_{21} = s_{0.30}$, $\hat{r}_{22} = s_1$, $\hat{r}_{23} = s_{4.28}$, $\hat{r}_{24} = s_{3.25}$, $\hat{r}_{25} = s_{0.31}$

$\hat{r}_{31} = s_{2.90}$, $\hat{r}_{32} = s_{0.23}$, $\hat{r}_{33} = s_1$, $\hat{r}_{34} = s_{4.29}$, $\hat{r}_{35} = s_{3.92}$

$\hat{r}_{41} = s_{0.33}$, $\hat{r}_{42} = s_{0.31}$, $\hat{r}_{43} = s_{0.23}$, $\hat{r}_{44} = s_1$, $\hat{r}_{45} = s_{4.37}$

$\hat{r}_{51} = s_{3.24}$, $\hat{r}_{52} = s_{3.27}$, $\hat{r}_{53} = s_{0.26}$, $\hat{r}_{54} = s_{0.23}$, $\hat{r}_{55} = s_1$

and thus, we get the collective multiplicative linguistic preference relation $\hat{R} = (\hat{r}_{ij})_{5\times5}$:

Table 4: The Collective Preference Relation $\hat{R}$

	x_1	x_2	x_3	x_4	x_5
x_1	s_1	$s_{3.37}$	$s_{0.35}$	s_3	$s_{0.31}$
x_2	$s_{0.30}$	s_1	$s_{4.28}$	$s_{3.25}$	$s_{0.31}$
x_3	$s_{2.90}$	$s_{0.23}$	s_1	$s_{4.29}$	$s_{3.92}$
x_4	$s_{0.33}$	$s_{0.31}$	$s_{0.23}$	s_1	$s_{4.37}$
x_5	$s_{3.24}$	$s_{3.27}$	$s_{0.26}$	$s_{0.23}$	s_1

Step 2: Utilize the EOWG operator (let its weighting vector be $w=(0.1, 0.2, 0.4, 0.2,0.1)^T$)

$\hat{r}_i = EOWG_w(\hat{r}_{i1}, \hat{r}_{i2}, \hat{r}_{i3}, \hat{r}_{i4}, \hat{r}_{i5})$ to aggregate $\hat{r}_{ij}(j = 1,2,3,4,5)$ corresponding to the alternative x_i, and then get the collective linguistic preference degree $\hat{r}_i(i = 1,2,3,4,5)$ of the i th alternative over all the other alternatives:

$$\hat{r}_1 = EOWG_w(\hat{r}_{11}, \hat{r}_{12}, \hat{r}_{13}, \hat{r}_{14}, \hat{r}_{15})$$

$$= (s_{3.37})^{0.1} \otimes (s_3)^{0.2} \otimes (s_1)^{0.4} \otimes (s_{0.35})^{0.2} \otimes (s_{0.31})^{0.1} = s_{1.01}$$

$$\hat{r}_2 = EOWG_w(\hat{r}_{21}, \hat{r}_{22}, \hat{r}_{23}, \hat{r}_{24}, \hat{r}_{25})$$

$$= (s_{4.28})^{0.1} \otimes (s_{3.25})^{0.2} \otimes (s_1)^{0.4} \otimes (s_{0.31})^{0.2} \otimes (s_{0.30})^{0.1} = s_{1.03}$$

$$\hat{r}_3 = EOWG_w(\hat{r}_{31}, \hat{r}_{32}, \hat{r}_{33}, \hat{r}_{34}, \hat{r}_{35})$$

$$= (s_{4.29})^{0.1} \otimes (s_{3.92})^{0.2} \otimes (s_{2.90})^{0.4} \otimes (s_1)^{0.2} \otimes (s_{0.23})^{0.1} = s_{2.01}$$

$$\hat{r}_4 = EOWG_w(\hat{r}_{41}, \hat{r}_{42}, \hat{r}_{43}, \hat{r}_{44}, \hat{r}_{45})$$

$$= (s_{4.37})^{0.1} \otimes (s_1)^{0.2} \otimes (s_{0.33})^{0.4} \otimes (s_{0.31})^{0.2} \otimes (s_{0.23})^{0.1} = s_{0.51}$$

$$\hat{r}_5 = EOWG_w(\hat{r}_{51}, \hat{r}_{52}, \hat{r}_{53}, \hat{r}_{54}, \hat{r}_{55})$$

$= (s_{3.27})^{0.1} \otimes (s_{3.24})^{0.2} \otimes (s_1)^{0.4} \otimes (s_{0.26})^{0.2} \otimes (s_{0.23})^{0.1} = s_{0.94}$

thus, we have $\hat{r}_3 > \hat{r}_2 > \hat{r}_1 > \hat{r}_5 > \hat{r}_4$

Step 3: Rank all the alternatives in accordance with the values of $\hat{r}_i \quad (i = 1,2,3,4,5)$:

$x_3 \succ x_2 \succ x_1 \succ x_5 \succ x_4$ and thus the most desirable alternative is x_3.

CONCLUSION

In this study, we have developed an extended induced ordered weighted geometric (EIOWG) operator, which takes as its argument pairs, called OWG pairs, in which one component is used to induce an ordering over the second components which are linguistic variables. We have studied some desirable properties of the EIOWG operator, and then applied the EIOWG operator to group decision making based on multiplicative linguistic preference relations. In the future, we shall continue working in the application of the EIOWG operator to other domains.

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