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Computation of the Locations of the Libration Points in the Relativistic Restricted Three-Body Problem

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Abstract: Problem statement: In this study, the effects of Relativistic Restricted three-Body Problem (in brief RRTBP) on the equilibrium points of both triangular and collinear is considered. The approximate locations of the collinear and triangular points are determined. Series expansions are used to develope in μ and $1/c^2$ as small parameters. To check the validity of our solution, when ignoring $1/c^2$ terms we get directly the classical results of the restricted three-body problem. **Conclusion/Recommendations:** A MATHEMATICA program is constructed to give a numerical application of the relativistic perturbations in the locations of the equilibrium points of the three body problem.

Key words: Relativistic restricted three body problem, location of the libration points, classical limit

INTRODUCTION

The three-body problem considers three mutually interacting masses m_1, m_2 and m_3. In the restricted three-body problem, m_3 is taken to be small enough so that it does not influence the motion of m_1 and m_2, which are assumed to be in circular orbits about their center of mass. The orbits of the three masses are further assumed to all lie in a common plane. If m_1 and m_2 are in elliptical instead of circular orbits, the problem is variously known as the "elliptic restricted problem" or "pseudo restricted problem" Szebehely (1967).

The history of the restricted three-body problem began with Euler and Lagrange continues with Jacobi (1836), Hill (1878), Poincare (1957) and Birkhoff (1915). Euler and Courvoisier (1980) first introduced a synodic (rotating) coordinate system, the use of which led to an integral of the equations of motion, known today as the Jacobian integral. Euler himself did not discover the Jacobi integral which was first given by Jacobi (1836) who, as Wintner remarks, "rediscovered" the synodic system. The actual situation is somewhat complex since Jacobi published his integral in a sidereal (fixed) system in which its significance is definitely less than in the synodic system.

Many authors hope to investigate the relativistic effects in this problem. But unfortunately, the Einstein field equations are nonlinear and therefore cannot in general be solved exactly. By imposing the symmetry requirements of time independence and spatial isotropy we are able to find one useful exact solution, the Schwarzschild metric, but we cannot actually make use of the full content of this solution, because in fact the solar system is not static and isotropic.

Indeed, the Newtonian effects of the planet's gravitational fields are an order of magnitude greater than the first corrections due to general relativity and completely swamp the higher corrections that are in principle provided by the exact Schwarzschild solution. It is worth noting to highlight some important articles in this field.

Computed the post-Newtonian deviations of the triangular Lagrangian points from their classical positions in a fixed frame of reference for the first time, but without explicitly stating the equations of motion. Treated the relativistic (RTBP) in rotating coordinates. He derived the Lagrangian of the system and the deviations of the triangular points as well. Weinberg (1972) calculated the components of the metric tensor by using the post-Newtonian approximation in order to obtain the (RTBP) problem equations of motion. Soffel (1989) obtained the angular frequency ω of the rotating frame for the relativistic two-body problem. Brumberg (1972; 1991) studied the problem in more details and collected most of the important results on relativistic celestial mechanics.

In this study, the approximate positions of the collinear and triangular points are determined using the

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equations of motion of the relativistic restricted three-body problem. The formulas are expanded in terms of μ and $\frac{1}{c^2}$ as small parameters

Equations of motion: The equations of motion of the infinitesimal mass in the (RRTBP) in a synodic frame of reference (ξ, η), in which the primary coordinates on the ξ -axis ($-\mu$, 0), ($1-\mu$, 0) are kept fixed and the origin at the center of mass, is given by Brumberg (1991), From Bhatnagar and Hallan (1997) Eq. 1:

$$\left.\begin{aligned} \ddot{\xi} - 2n\dot{\eta} &= \frac{\partial U}{\partial \xi} - \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{\xi}}\right) \\ \ddot{\eta} + 2n\dot{\xi} &= \frac{\partial U}{\partial \eta} - \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{\eta}}\right) \end{aligned}\right\} \quad (1)$$

where, U is the potential-like function of the (RRTBP), which can be written as composed of two components, namely the potential of the classical (RTBP) potential U_c and the relativistic correction U_r Eq. 2:

$$U = U_c + U_r \quad (2)$$

where, U_c and U_r are given by Eq. 3:

$$U_c = \frac{r^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \quad (3)$$

And Eq. 4:

$$\begin{aligned} U_r = {} & \frac{r^2}{2c^2}(\mu(1-\mu)-3) + \frac{1}{8c^2}((\xi+\dot{\eta})^2 \\ & +(\eta-\dot{\xi})^2)^2 + \frac{3}{2c^2}\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right) \\ & \times((\xi+\dot{\eta})^2+(\eta-\dot{\xi})^2) - \frac{1}{2c^2}\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right)^2 \\ & -\frac{\mu(1-\mu)}{2c^2}\left[\begin{matrix} \frac{1}{r_1}+\left(\frac{1}{r_1}-\frac{1}{r_2}\right)(1-3\mu-7\xi-8\dot{\eta})+\eta^2 \\ \left(\frac{\mu}{r_1^3}+\frac{1-\mu}{r_2^3}\right) \end{matrix}\right] \end{aligned} \quad (4)$$

with:

$$\left.\begin{aligned} n &= 1+\frac{1}{2c^2}(\mu(1-\mu)-3), & r &= \sqrt{(\xi^2+\eta^2)} \\ r_1 &= \sqrt{(\xi+\mu)^2+\eta^2}, & r_2 &= \sqrt{(\xi+\mu-1)^2+\eta^2} \end{aligned}\right\}$$

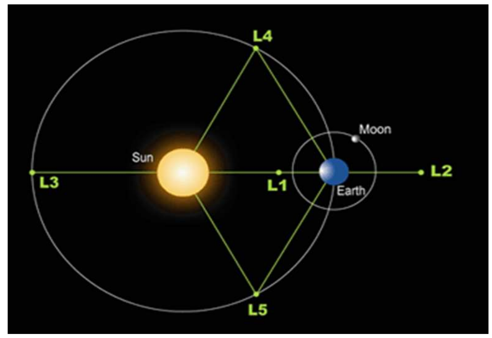

Fig. 1: The five Lagrangian points with (Sun-Earth system; as an example)

Location of the libration points: From the equations of motion (1), it is apparent that an equilibrium solution exists relative to the rotating frame when the partial derivatives of the pseudopotential functions (U_ξ, U_η, U_ζ) are all zero, i.e., U = const. These points correspond to the positions in the rotating frame at which the gravitational and the centrifugal forces associated with the rotation of the synodic frame all cancel, with the result that a particle located at one of these points appears stationary in the synodic frame. There are five equilibrium points in the circular (RTBP), also known as Lagrange points or libration points. Three of them (the collinear points) lie along the ξ-axis: one interior point between the two primaries and one point on the far side of each primary with respect to the barycenter. The other two libration points (the triangular points) are each located at the apex of an equilateral triangle formed with the primaries. One of the most commonly used nomenclatures (and the one that we will use here) defines the interior points as L_1, the point exterior to the small primary (the planet) as L_2 and the point L_3 as exterior to the large primary (the Sun), see Fig. 1. The triangular points are designated L_4 and L_5, with L_4 moving ahead of the ξ-axis and L_5 trailing ξ-axis, along the orbit of the small primary as the synodic frame rotates uniformly relative to the inertial frame (as shown in Fig. 1)

Remark: All five libration points lie in the ξ-η plane, i.e., in the plane of motion

The libration points are obtained from equations of motion after setting $\ddot{\xi} = \ddot{\eta} = \ddot{\zeta} = 0$. These points represent particular solutions of equations of motion Eq. 5:

$$\frac{\partial U}{\partial \xi} = \frac{\partial U_c}{\partial \xi} + \frac{\partial U_r}{\partial \xi} = 0, \quad \frac{\partial U}{\partial \eta} = \frac{\partial U_c}{\partial \eta} + \frac{\partial U_r}{\partial \eta} = 0 \quad (5)$$

The explicit formulas are (Remembering that U = $U_c + U_r$) Eq. 6:

$$
\begin{aligned}
\frac{\partial U}{\partial \xi} &= \xi - \frac{(1-\mu)(\xi+\mu)}{r_1^3} - \frac{\mu(\xi+\mu-1)}{r_2^3} \\
&+ \frac{1}{c^2}\Bigg\{(\mu-\mu^2-3)\xi + \left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right) \\
&\times\left(\frac{(1-\mu)(\xi+\mu)}{r_1^3}+\frac{\mu(\xi+\mu-1)}{r_2^3}\right)+\frac{1}{2}(\eta^2+\xi^2)\xi \\
&-\frac{3}{2}\left(\frac{(1-\mu)(\xi+\mu)}{r_1^3}+\frac{\mu(\xi+\mu-1)}{r_2^3}\right)(\eta^2+\xi^2) \\
&+3\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right)-\frac{1}{2}\mu(1-\mu)\Bigg[-\frac{1}{r_1^3}(\xi+\mu) \\
&+\left(\frac{(\xi+\mu)}{r_1^3}+\frac{(\xi+\mu-1)}{r_2^3}\right)(-1+3\mu+7\xi) \\
&-7\left(\frac{1}{r_1}-\frac{1}{r_2}\right)-3\eta^2\left(\frac{\mu(\xi+\mu)}{r_1^5}+\frac{(1-\mu)(\xi+\mu-1)}{r_2^5}\right)\Bigg]\Bigg\}=0
\end{aligned}
\tag{6}
$$

And Eq. 7:

$$
\begin{aligned}
\frac{\partial U}{\partial \eta} &= \eta - \frac{(1-\mu)}{r_1^3}\eta - \frac{\mu}{r_2^3}\eta + \frac{1}{c^2} \\
&\times\Bigg\{(\mu-\mu^2-3)\eta+\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right)\left(\frac{(1-\mu)}{r_1^3}+\frac{\mu}{r_2^3}\right)\eta \\
&+\frac{1}{2}(\xi^2+\eta^2)\eta-\frac{3}{2}(\xi^2+\eta^2)\left(\frac{(1-\mu)}{r_1^3}+\frac{\mu}{r_2^3}\right)\eta \\
&+3\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right)\eta+\frac{\mu(1-\mu)}{2}\eta+\frac{1}{2}\mu(1-\mu)\eta \\
&\times\Bigg[\left(-\frac{1}{r_1^3}+\frac{1}{r_2^3}\right)(-1+3\mu+7\xi)+3\left(\frac{(1-\mu)}{r_2^5}+\frac{\mu}{r_1^5}\right)\eta^2 \\
&+\frac{1}{r_1^3}-2\left(\frac{\mu}{r_1^3}+\frac{1-\mu}{r_2^3}\right)\Bigg]\Bigg\}=0
\end{aligned}
\tag{7}
$$

Location of collinear libration points:

Location of L_1.

The collinear points must, by definition, have $\xi = \eta = 0$, since $1/c^2 \ll 1$ and the solution of the classical (RTBP) is (Fig. 2) Eq. 8:

$$
\begin{aligned}
&r_1 + r_2 = 1, \qquad r_1 = \xi+\mu, \\
&r_2 = 1-\mu-\xi, \qquad \frac{\partial r_1}{\partial \xi} = -\frac{\partial r_2}{\partial \xi} = 1
\end{aligned}
\tag{8}
$$

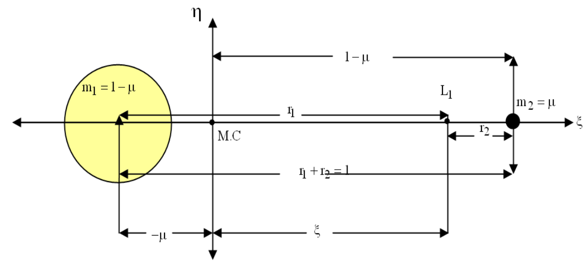

Fig. 2: Shows the location of L_1

Using (8) Eq. 6, can be written explicitly in terms of r_1 and r_2 as Eq. 9:

$$
\begin{aligned}
\frac{\partial U}{\partial \xi} &= 1-\mu-r_2-\left(\frac{1-\mu}{r_1^2}-\frac{\mu}{r_2^2}\right) \\
&+\frac{1}{c^2}\Bigg\{\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right)\left(\frac{1-\mu}{r_1^2}-\frac{\mu}{r_2^2}\right) \\
&+(\mu-\mu^2-3)(1-\mu-r_2)+\frac{1}{2}(1-\mu-r_2)^3 \\
&-\frac{3}{2}\left(\frac{1-\mu}{r_1^2}-\frac{\mu}{r_2^2}\right)(1-\mu-r_2)^2+3\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right) \\
&\times(1-\mu-r_2)-\frac{1}{2}\mu(1-\mu)\Bigg[-\frac{1}{r_1^2}+\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}\right)(-1+3\mu \\
&+7(1-\mu-r_2))-7\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\Bigg]\Bigg\}=0
\end{aligned}
\tag{9}
$$

Then it may be reasonable in our case to assume that position of the equilibrium points L_1 are the same as given by classical (RTBP) but perturbed due to the inclusion of the relativistic correction by quantities ($\varepsilon_1 \equiv O(1/c^2)$) Eq. 10:

$$r_1 = a_1+\varepsilon_1,\ r_2 = b_1 - \varepsilon_1,\ a_1 = 1- b_1 \tag{10}$$

where, a_1 and b_1 are unperturbed positions of r_1 and r_2 respectively and b_1 is given after some successive approximation by:

$$b_1 = \alpha\left(1-\frac{\alpha}{3}-\frac{\alpha^2}{9}+\frac{2}{27}\alpha^3+\frac{2}{81}\alpha^4\right);\ \ \alpha = \left(\frac{\mu}{3(1-\mu)}\right)^3$$

Substituting from Eq. 10 into Eq. 9 and retaining the terms up to the first order in the small parameter ε_1, we get:

$$
\begin{aligned}
\frac{\partial U}{\partial \xi} &= 1-\mu-b_1+\varepsilon_1-\frac{1-\mu}{(1-b_1)^2}\left(1-\frac{2\varepsilon_1}{1-b_1}\right)\\
&+\frac{\mu}{b_1^2}\left(1+\frac{2\varepsilon_1}{b_1}\right)+\frac{1}{c^2}\left\{\begin{aligned}&(\mu-\mu^2-3)\,(1-\mu-b_1)\\ &+\left(\frac{1-\mu}{(1-b_1)}+\frac{\mu}{b_1}\right)\end{aligned}\right.\\
&\times\left(\frac{1-\mu}{(1-b_1)^2}-\frac{\mu}{b_1^2}\right)+\frac{1}{2}(1-\mu-b_1)^3\\
&-\frac{3}{2}\left(\frac{1-\mu}{(1-b_1)^2}-\frac{\mu}{b_1^2}\right)(1-\mu-b_1)^2\\
&+3\left(\frac{1-\mu}{(1-b_1)}+\frac{\mu}{b_1}\right)(1-\mu-b_{11})\\
&-\frac{1}{2}\mu\,(1-\mu)\left[-\frac{1}{(1-b_1)^2}-7\left(\frac{1}{(1-b_1)}-\frac{1}{b_1}\right)\right.\\
&\left.\left.+\left(\frac{1}{(1-b_1)^2}+\frac{1}{b_1^2}\right)(6-7b_1-4\mu)\right]\right\}=0
\end{aligned}
\tag{11}
$$

Setting,

$$
\left(1+\frac{2(1-\mu)}{(1-b_1)^3}+\frac{2\mu}{b_1^3}\right)^{-1}=d_1,\ \frac{1}{b_1}=e_1,
$$

$$
\frac{1}{b_1^2}=f_1,\ \frac{1}{1-b_1}=g_1,\ \frac{1}{(1-b_1)^2}=h_1
$$

Eq. 11 -13 can be solved for ε_1 to yield:

$$
\begin{aligned}
\varepsilon_1 &= d_1\,\big(-1+b_1+\mu+(1-\mu)\,h_1-\mu f_1\\
&-\frac{1}{c^2}\Big\{(\mu-\mu^2-3)\,(1-\mu-b_1)\\
&+\big((1-\mu)g_1+\mu e_1\big)\big((1-\mu)h_1-\mu f_1\big)\\
&+\frac{1}{2}(1-\mu-b_1)^3-\frac{3}{2}\big((1-\mu)h_1-\mu f_1\big)(1-\mu-b_1)^2\\
&+3\big((1-\mu)g_1+\mu e_1\big)(1-\mu-b_1)\\
&-\frac{1}{2}\mu\,(1-\mu)\left[\begin{aligned}&-h_1-7(g_1-e_1)\\ &+(h_1+f_1)(6-7b_1-4\mu)\end{aligned}\right]\Big\}
\end{aligned}
\tag{12}
$$

Expanding b_1, d_1, e_1, f_1, g_1 and h_1 in order of μ retaining terms up to order $(\frac{\mu}{3})^3$. Then substituting back into Eq.s 10 and 8 we get the location for the first libration point $\xi_{o,L1}$

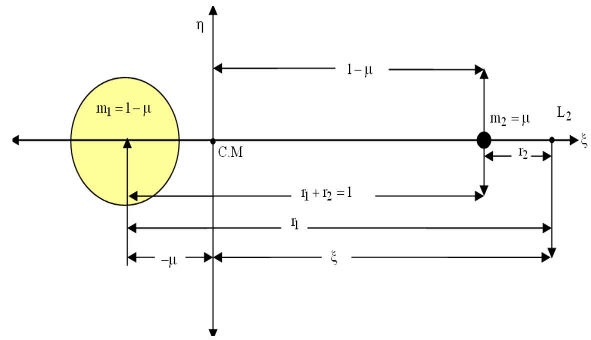

Fig. 3: Shows the Location of L_2

$$
\begin{aligned}
\xi_{0,L_1} &= 1-\{(\frac{\mu}{3})^{\frac{1}{3}}+\frac{1}{3}(\frac{\mu}{3})^{\frac{2}{3}}-\frac{26}{9}(\frac{\mu}{3})-\\
&\frac{58}{81}(\frac{\mu}{3})^{\frac{4}{3}}+\frac{11}{243}(\frac{\mu}{3})^{\frac{5}{3}}-\frac{4}{9}(\frac{\mu}{3})^2\\
&-\frac{3544}{6561}(\frac{\mu}{3})^{\frac{7}{3}}-\frac{34030}{6561}(\frac{\mu}{3})^{\frac{8}{3}}-\frac{2561}{729}(\frac{\mu}{3})^3+\ldots\}\\
&-\frac{1}{c^2}\{-\frac{1}{3}(\frac{\mu}{3})^{\frac{1}{3}}+\frac{5}{9}(\frac{\mu}{3})^{\frac{2}{3}}-\frac{4}{3}(\frac{\mu}{3})+\frac{2425}{486}(\frac{\mu}{3})^{\frac{4}{3}}\\
&-\frac{1729}{486}(\frac{\mu}{3})^{\frac{5}{3}}-\frac{6395}{2187}(\frac{\mu}{3})^2+\frac{398335}{336366}(\frac{\mu}{3})^{\frac{7}{3}}\\
&+\frac{422957}{59049}(\frac{\mu}{3})^{\frac{8}{3}}-\frac{8374501}{354294}(\frac{\mu}{3})^3+\ldots\}
\end{aligned}
\tag{13}
$$

In this equation terms that are not factored by $1/c^2$ represent the location of $\xi_{o,L1}$ in the classical RTBP while the terms that factored by $1/c^2$ represent the correction due to the inclusion of the relativistic terms.

Location of L_2: The L_2 point locates outside the small massive primary of mass μ. We now drive an approximate location for this point. At the L_2 point we have (see Fig. 3) Eq. 14:

$$
\begin{aligned}
&r_1-r_2=1, &&r_1=\xi+\mu,\\
&r_2=\xi+\mu-1, &&\frac{\partial r_1}{\partial \xi}=\frac{\partial r_2}{\partial \xi}=1
\end{aligned}
\tag{14}
$$

The procedure is similar to Eq. 10 with little modification according to the location of $\xi_{o,L2}$ as:

$$
r_1=a_2+\varepsilon_2,\qquad r_2=b_2+\varepsilon_2,\qquad a_2=1+b_2
$$

With a_2 and b_2 are the unperturbed positions of r_1 and r_2 respectively, where b_2 is by:

$$b_2 = \alpha\left(1+\frac{\alpha}{3}-\frac{\alpha^2}{9}-\frac{2}{27}\alpha^3+\frac{2}{81}\alpha^4\right),$$

$$\alpha = \left(\frac{\mu}{3(1-\mu)}\right)^3$$

As is done in the calculation of the location of $\xi_{o,L1}$ we can similarly calculate the location of $\xi_{o,L2}$ as Eq. 15:

$$\begin{aligned}
\xi_{0,L_2} &= 1+\{(\frac{\mu}{3})^{\frac{1}{3}}+\frac{1}{3}(\frac{\mu}{3})^{\frac{2}{3}}-\frac{28}{9}(\frac{\mu}{3}) \\
&+\frac{50}{81}(\frac{\mu}{3})^{\frac{4}{3}}+\frac{34}{243}(\frac{\mu}{3})^{\frac{5}{3}}-\frac{40}{81}(\frac{\mu}{3})^2 \\
&+\frac{5542}{6561}(\frac{\mu}{3})^{\frac{7}{3}}-\frac{7505}{2187}(\frac{\mu}{3})^{\frac{8}{3}}+\frac{162625}{59049}(\frac{\mu}{3})^3+\ldots\} \\
&+\frac{1}{c^2}\{-\frac{1}{3}(\frac{\mu}{3})^{\frac{1}{3}}-\frac{5}{9}(\frac{\mu}{3})^{\frac{2}{3}}-\frac{4}{3}(\frac{\mu}{3}) \\
&+\frac{1084}{486}(\frac{\mu}{3})^{\frac{4}{3}}+\frac{886}{243}(\frac{\mu}{3})^{\frac{5}{3}}-\frac{8843}{2178}(\frac{\mu}{3})^2 \\
&-\frac{12796}{19683}(\frac{\mu}{3})^{\frac{7}{3}}+\frac{2872}{59049}(\frac{\mu}{3})^{\frac{8}{3}}-\frac{1915435}{354294}(\frac{\mu}{3})^3+\ldots\}
\end{aligned} \tag{15}$$

Location of L_3: The L_3 point lies on the negative ξ-axis. We now derive an approximate location for this point. At the L_3 point we have (Fig. 4) Eq. 16:

$$\begin{aligned}
&r_2 - r_1 = 1, \quad r_1 = -\xi - \mu, \\
&r_2 = 1-\mu-\xi, \quad \frac{\partial r_1}{\partial \xi} = \frac{\partial r_2}{\partial \xi} = -1
\end{aligned} \tag{16}$$

The procedure is similar to Eq. 10 with little modification according to the location of $\xi_{o,L2}$ as:

$$r_1 = a_3 + \varepsilon_3, \quad r_2 = b_3 + \varepsilon_3, \quad a_3 = b_3 - 1$$

where a_3 and b_3 are the unperturbed values of r_1 and r_2 respectively and b_3 is given after some successive approximation by:

$$b_3 = 2 - \frac{7}{12}\mu\left(1+\frac{23}{144}\mu+\frac{25921}{2985984}\mu^4\right)$$

As is done in the calculation of the location of $\xi_{o,L1}$, we can similarly calculate the location of $\xi_{o,L3}$ as Eq. 17:

$$\begin{aligned}
\xi_{0,L_3} &= \{-1-\frac{5}{12}\mu-\frac{1127}{20736}\mu^3-\frac{7889}{248832}\mu^4 \\
&-\frac{698005}{3981312}\mu^5-\ldots\}-\frac{1}{c^2}\{-\frac{3}{4}\mu+\frac{7}{16}\mu^2 \\
&+\frac{3227}{41472}\mu^3+\frac{51037}{497664}\mu^5+\ldots\}
\end{aligned} \tag{17}$$

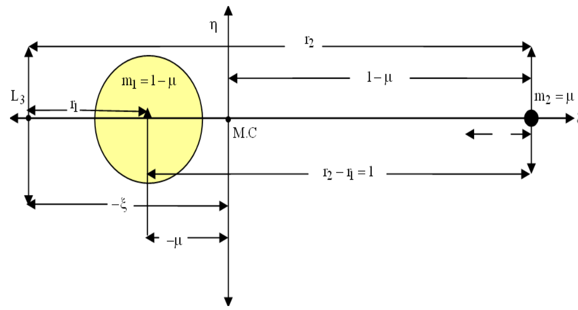

Fig. 4: Shows the Location of L_3

Location of the Triangular Points L4,5: Since $1/c^2 \ll 1$ and the solution of the classical restricted three-body problem is $r_1 = r_2 = 1$, then it may be reasonable in our case to assume that the positions of the equilibrium points $L_{4,5}$ are the same as given by classical restricted three-body problem but perturbed due to the inclusion of the relativistic correction by quantities ($\varepsilon_{1,2} \equiv O\left(1/c^2\right)$)Eq. 18:

$$r_1 = (1+\varepsilon_1), \quad r_2 = (1+\varepsilon_2) \tag{18}$$

Substituting in the second set of (7) and (8) and solving for ξ and η up to the first order in the involved small quantities ε_1 and ε_2, we get:

$$\left.\begin{aligned}
&\xi = (\varepsilon_1 - \varepsilon_2 + \frac{1-2\mu}{2}) \\
&\eta = \pm\left[\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{3}(\varepsilon_1+\varepsilon_2)\right].
\end{aligned}\right\} \tag{19}$$

Substituting the values of r_1, r_2, ξ and η into equations:

$$\frac{\partial U}{\partial \xi}=\frac{\partial U_c}{\partial \xi}+\frac{\partial U_r}{\partial \xi}=0\,, \quad \frac{\partial U}{\partial \eta}=\frac{\partial U_c}{\partial \eta}+\frac{\partial U_r}{\partial \eta}=0$$

Evaluating the included partial derivatives and retaining the terms up to the first order in the small parameters ε_1, ε_2 and also the first order terms in the relativistic correction, we obtained Eq. 20:

$$\left.\begin{aligned}
&(1-\mu)\varepsilon_1 - \mu\varepsilon_2 - \frac{3}{8c^2}\mu(1-2\mu) \\
&(1-\mu) = 0, (1-\mu)\varepsilon_1 + \mu\varepsilon_2 \\
&-\frac{7}{8c^2}\mu\,(1-\mu) \qquad = 0.
\end{aligned}\right\} \tag{20}$$

Which represent two simultaneous equations in ε_1 and ε_2. Their solutions are being Eq. 21:

$$\left.\begin{aligned} \varepsilon_1 &= -\frac{1}{8c^2}\mu(2+3\mu) \\ \varepsilon_2 &= -\frac{1}{8c^2}(1-\mu)(5-3\mu) \end{aligned}\right\} \tag{21}$$

Substituting the values of ε_1 and ε_2 into Eq. 19, yields the coordinates of the triangular points Eq. 22:

$$\left.\begin{aligned} \xi_{0,L_{4,5}} &= \frac{(1-2\mu)}{2}\left(1+\frac{5}{4c^2}\right) \\ \eta_{0,L_{4,5}} &= \pm\frac{\sqrt{3}}{2}\left(1-\frac{1}{12c^2}(6\mu^2-6\mu+5)\right) \end{aligned}\right\} \tag{22}$$

The classical limit: To check our solution, when ignoring $1/c^2$ terms we get directly the classical results of the restricted three-body problem as:

$$\xi_{0,L_1} = 1-\left\{\begin{aligned} &\left(\frac{\mu}{3}\right)^{\frac{1}{3}}+\frac{1}{3}\left(\frac{\mu}{3}\right)^{\frac{2}{3}}-\frac{26}{9}\left(\frac{\mu}{3}\right)-\frac{58}{81} \\ &\left(\frac{\mu}{3}\right)^{\frac{4}{3}}+\frac{11}{243}\left(\frac{\mu}{3}\right)^{\frac{5}{3}}-\frac{4}{9}\left(\frac{\mu}{3}\right)^2 \\ &-\frac{3544}{6561}\left(\frac{\mu}{3}\right)^{\frac{7}{3}}-\frac{34030}{6561}\left(\frac{\mu}{3}\right)^{\frac{8}{3}}-\frac{2561}{729}\left(\frac{\mu}{3}\right)^3+\ldots \end{aligned}\right\}$$

$$\begin{aligned} \xi_{0,L_2} = \{&1+\left(\frac{\mu}{3}\right)^{\frac{1}{3}}+\frac{1}{3}\left(\frac{\mu}{3}\right)^{\frac{2}{3}}-\frac{28}{9}\left(\frac{\mu}{3}\right) \\ &+\frac{50}{81}\left(\frac{\mu}{3}\right)^{\frac{4}{3}}+\frac{34}{243}\left(\frac{\mu}{3}\right)^{\frac{5}{3}}-\frac{40}{81}\left(\frac{\mu}{3}\right)^2+\frac{5542}{6561}\left(\frac{\mu}{3}\right)^{\frac{7}{3}} \\ &-\frac{7505}{2187}\left(\frac{\mu}{3}\right)^{\frac{8}{3}}+\frac{162625}{59049}\left(\frac{\mu}{3}\right)^3+\ldots\} \end{aligned}$$

$$\begin{aligned} \xi_{0,L_3} &= 1-\frac{5}{12}\mu-\frac{1127}{20736}\mu^3 \\ &-\frac{7889}{248832}\mu^4-\frac{698005}{3981312}\mu^5-\ldots \end{aligned}$$

$$\xi_{0,L_{4,5}} = \frac{(1-2\mu)}{2},\ \eta_{0,L_{4,5}} = \pm\frac{\sqrt{3}}{2}$$

Numerical results and concluding remarks: We have used the above mentioned analysis and the explicit formulas obtained to design a computer program using MATHEMATICA software. We have plotted the locations of the equilibrium points L_1, L_2, L_3, versus the whole range of the mass ratio $\mu \in [0, 0.5]$.

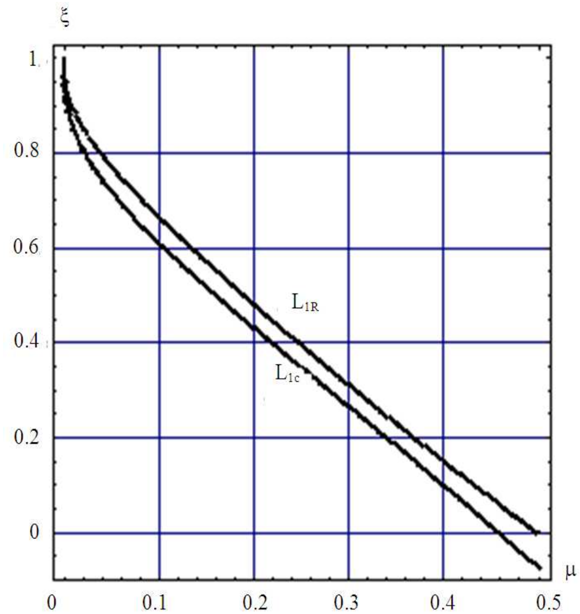

Fig. 5: Location of the Equilibrium point L_1 versus the mass ratio μ.

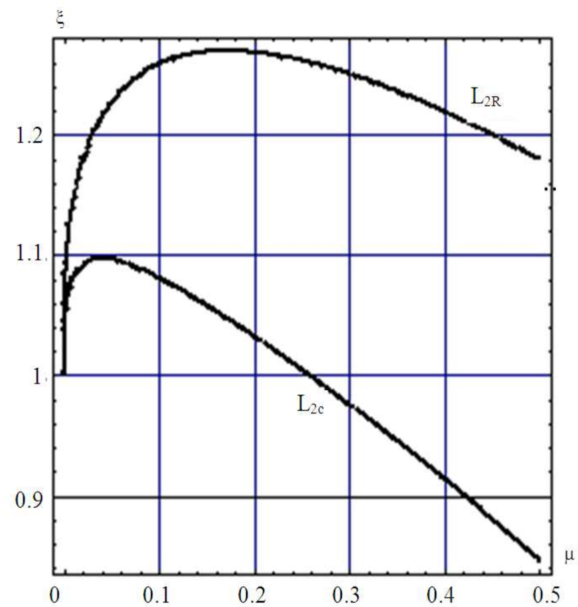

Fig. 6: Location of the Equilibrium point L_2 versus the mass ratio μ.

In Fig. 5-8, [L_{ic}, i = 1, 2, 3] denotes for the position of the equilibrium points without the relativistic contribution, i.e within the classical problem of the restricted three bodies, while [L_{iR}, i = 1, 2, 3] denotes for the position of the equilibrium points including the relativistic contribution.

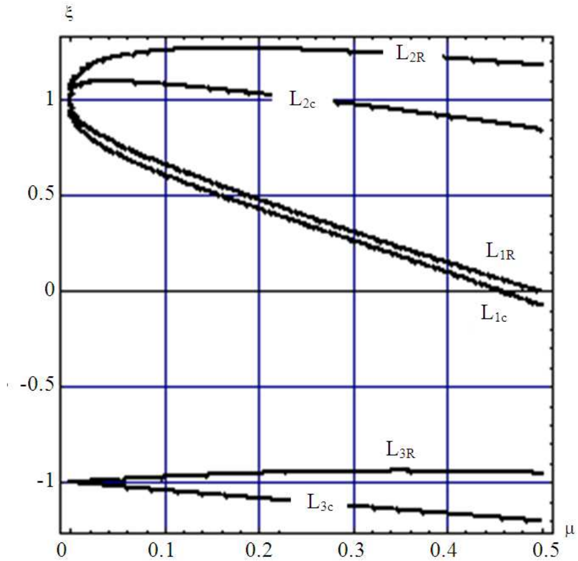

Fig. 7: Location of the Equilibrium point L_3 versus the mass ratioμ

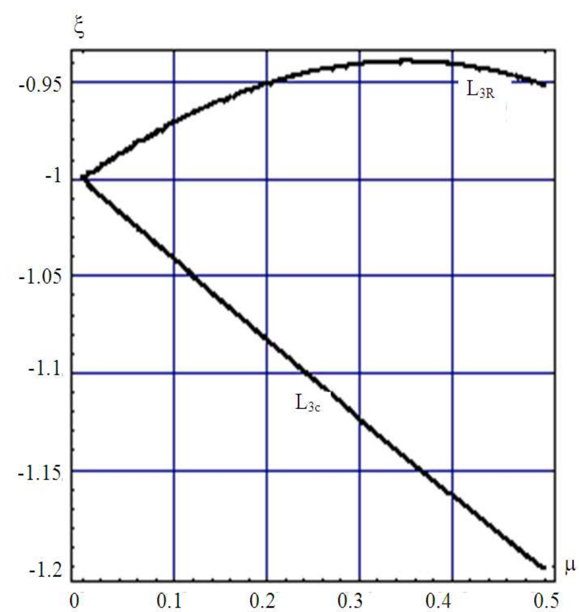

Fig. 8: Location of the Equilibrium points L_1, L_2, L_3 versus the mass ratioμ .

In Fig. 5 the difference L_{iR}-L_{1c} seems to be constant beyond $\mu \geq 0.05$. In Fig. 6 the difference L_{2R} - L_{2c} increases with increasing the mass ratioμ. In Fig. 7 the difference L_{3R} - L_{3c} increases with increasing the mass ratio μ within the domain $0 \leq \mu \leq 0.35$ and decreases with increasing μ within the domain $0.35 \leq \mu \leq 0.5$. Fig. 8 represents an assembly plot of the all collinear equilibrium points.

We may see that, in most of these cases, the positions of L_3 and L_2 are much more affected by the influence of the relativistic terms than that of the equilibrium point, L_1

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