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Approximation Methods for Fixed Points of Quasi-contractive Mappings in Arbitrary Real Banach Spaces

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Abstract: Owojori and Imoru [1] introduced some iteration methods and investigated its convergence for pseudocontractive and accretive operators in arbitrary Banach spaces. In this work, strong convergence results are established for fixed points of a general type of quasicontractive mappings in arbitrary real Banach spaces. Our results here represent some improvements on the earlier results of Owojori and Imoru [1], Naimpally and Singh [2], Chidume [3, 4], Qihou [5], Ganguly and Bandyopahyay [6], Chidume and Osilike [7], Rhoades [8] and Osilike [9] for quasicontractive operators in Banach spaces.

Key words: Quasicontractive Mappings, Arbitrary Real Banach Spaces, Iteration Methods

INTRODUCTION

Let K be a nonempty subset of an arbitrary real Banach space X and T a selfmapping of K. An operator $T: K \to K$ is called quasi-contractive, in the general sense, if the following inequality:

$$T_x - T_y \leq s_1 \; x - y \; + s_2[\; x - T_x \; + \; y - T_y \;] + s_3[\; x - T_y \; + \; y - T_x \;] \quad (1.1)$$

holds for all $x,y \in K$, where s_1, s_2, s_3 are real constants in [0,1] satisfying $s_1 + 2s_2 + 2s_3 = 1$.

Several authors including the authors, Rhoades [8], Chidume [3, 4], Osilike [9], Chidume and Osilike [7], Quhou [5] among others, have investigated the fixed points of quasi-contractive mappings by the Mann and Ishikawa iteration methods (including their modifications) and fixed point results have been established.

Recently, Owojori and Imoru [1] introduced a more acceptable three-step iteration method which contains the previous iteration methods of Mann and Ishikawa as special cases. It is defined for arbitrary $x_1 \in K$ – a closed bounded convex subset of a Banach space B, by:

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T y_n + c_n S x_n \\ y_n &= a'_n x_n + b'_n S z_n + c'_n v_n \\ z_n &= a''_n x_n + b'''_n S z_n + c''_n w_n \end{aligned}\right\} n \geq 1 \quad (1.2)$$

where, S,T are nonlinear uniformly continuous self-mappings of K satisfying some contractive definitions and $\{v_n\}$, $\{w_n\}$ are bounded sequences in K. Also $\{a_n\}$, $\{a'_n\}$, $\{a''_n\}$, $\{b_n\}$, $\{b'_n\}$, $\{b''_n\}$, $\{c_n\}$, $\{c'_n\}$, $\{c''_n\}$, are real sequences in [0,1] satisfying:

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1$,

(ii) $\sum b_n = \infty$

Two special cases of (1.2) are given respectively by:

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T y_n + c_n T x_n \\ y_n &= a'_n x_n + b'_n T z_n + c'_n v_n \\ z_n &= a''_n x_n + b'''_n T x_n + c''_n w_n \end{aligned}\right\} n \geq 1 \quad (1.3)$$

and

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T y_n + c_n u_n \\ y_n &= a'_n x_n + b'_n T z_n + c'_n v_n \\ z_n &= a''_n x_n + b'''_n T x_n + c''_n w_n \end{aligned}\right\} n \geq 1 \quad (1.4)$$

where, the parameters satisfy the same conditions as for (1.2).

We observed that the iteration schemes given by (1.2), (1.3) and (1.4) are all well defined and are generalizations of the Mann and Ishikawa titration schemes (with or without errors). Also, (1.3) is a slight generalization of (1.4). Our purpose in this manuscript is to establish the convergence of the iteration methods (1.4), (1.3) and (1.2) to the fixed points and common fixed points of the general quasi-contractive operator given by (1.1), in arbitrary real Banach space.

RESULTS

In the sequel, we shall require the following result due to Weng [10].

Lemma 2.1: Let $\{\rho_n\}$ be a nonnegative sequence of real numbers satisfying:

$\rho_{n+1} \le (1-\delta_n)\rho_n + \sigma_n$ (2.1)

where $\delta_n \in [0,1]$, $\sum\delta_i = \infty$, and $\sigma_n = o(\delta_n)$. Then $\lim_{n\to\infty}\rho_n = o.$

Theorem 2.2: Let X be an arbitrary real Banach space and K a closed, convex, bounded subset of X. Suppose T is a uniformly continuous selfmapping of K satisfying the quasi-contractive definition (1.1). Define sequence $\{x_n\}$ iteratively for arbitrary $x_1 \in K$ by:

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T y_n + c_n u_n \\ y_n &= a'_n x_n + b'_n T z_n + c'_n v_n \\ z_n &= a''_n x_n + b'''_n T x_n + c''_n w_n \end{aligned}\right\} \quad n \ge 1$$

where $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are arbitrary sequences in K and $\{a_n\}$, $\{a'_n\}$, $\{a''_n\}$, $\{b_n\}$, $\{b'_n\}$, $\{b''_n\}$, $\{b''_n\}$, $\{c_n\}$, $\{c'_n\}$, $\{c''_n\}$, are real sequences in [0,1] satisfying:

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1$,

(ii) $\sum b_n = \infty$

(iii) $a_n := b_n + c_{n\beta n} := b'_n + c'_n \Upsilon_n := b''_n + c''_n$ and $k_1 \le \dfrac{1-\alpha_n}{\alpha_n\beta_n\gamma_n}$

(iv) $\dfrac{s_1 - s_3}{1-s_3} < 1$

Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof: Since T is uniformly continuous on the closed bounded convex set K, then, by Deimling [11], F(T) is nonempty. Let $x^* \in F(T)$. Then, from our hypothesis.

$$\begin{aligned} x_{n+1}-x^* &= a_n x_n + b_n T y_n + c_n u_n - x^* \\ &= (1-\alpha_n)(x_n - x^*) + \alpha_n (T y_n - x^*) \\ &\quad + c_n(u_n - T y_n) \\ &\le (1-\alpha_n) \| x_n - x^* \| + \alpha_n \| T y_n - x^* \| \\ &\quad + c_n \| u_n - T y_n \| \end{aligned} \tag{2.2}$$

Since T is quasi-contractive in the sense of (1.1), then

$$\|Ty_n - x^*\| \le s_1 \|y_n - x^*\| + s_2[\|y_n - Ty_n\| + \|x^* - x^*\|] + s_3[\|y_n - x^*\| + \|x^* - Ty_n\|]$$

Therefore,

$$(1-s_3)\|Ty_n - x^*\| \le (s_1 + s_3)\|y_n - x^*\| + s_2\|y_n - Ty_n\|$$

Therefore,

$$\begin{aligned} \|Ty_n - x^*\| &\le \frac{s_1 - s_3}{1-s_3}\|y_n - x^*\| \\ &\quad + \frac{s_2}{1-s_3}\|y_n - Ty_n\| \\ &= k_1\|y_n - x^*\| + k_2\|y_n - Ty_n\| \end{aligned} \tag{2.3}$$

where, $k_1 = \dfrac{s_1 - s_3}{1-s_3} < 1$ and $k_2 = \dfrac{s_2}{1-s_3} < 1$

Continuity of T implies that there exists a real number $M_1 < \infty$ such that:

$$\|u_n - Ty_n\| \le M_1, \text{ and } \|y_n - Ty_n\| \le M_1.$$

Also $c_n \le \alpha_n$. Substituting into (2.3), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n k_1)\|y_n - x^*_n\| \\ &\quad + \alpha_n(k_2+1)M_1 \end{aligned} \tag{2.4}$$

We also have the following estimates:

$$\begin{aligned} y_n - x^* &= a_n x_n + b_n T z_n + c'_n v_n - x^* \\ &\le (1-\beta_n)(x_n - x^*) + \beta_n(Tz_n - x^*) + c'_n(v_n - Tz_n) \quad (2.5) \\ &\le (1-\beta_n)\|x_n - x^*)\| + \beta_n\|Tz_n - x^*\| + c'_n\|v_n - Tz_n\| \end{aligned}$$

By similar estimates as above, we also have:

$$\|Tz_n - x^*\| \le k_1\|z_n - x^*\| + k_2\|z_n - Tz_n\| \tag{2.6}$$

Continuity of T implies that there exists a real number $M_2 < \infty$ such that:

$$\|Tz_n - v_n\| \le M_2 \text{ and } \|z_n - Tz_n\| \le M_2$$

Therefore, substituting (2.6) into (2.5), we have:

$$\begin{aligned} \|y_n - x^*\| &\le (1-\beta_n)\|x_n - x^*\| + \beta_n k_1\|z_n - x^*\| \\ &\quad + \beta_n k_2 M_2 + \beta_n M_2 \end{aligned} \tag{2.7}$$

Substitute (2.7) into (2.4), we obtain:

$$\begin{aligned} \|x_{n+1} - x^*\| &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n k_1(1-\beta_n)\|x_n - x^*\| \\ &= \beta_n k_1\|z_n - x^*\| + \beta_n k_2 M_2 + \beta_n M_2] + \alpha_n(k_2+1)M_1 \quad (2.8) \\ &\le [1-\alpha_n(1-k_1(1-\beta_n))]\|x_n - x^*\| + \alpha_n\beta_n k_1\|z_n - x^*\| \\ &\quad + \alpha_n\beta_n k_1(k_2+1)M_2 + \alpha_n(k_2+1)M_1 \end{aligned}$$

We also have the following estimates:

$$\|z_n x^*\| \le [1-\Upsilon_n(1-k_1)]\|x_n - x^*\| + \Upsilon_n(k_2+1)M_3 \tag{2.9}$$

where, $M_3 < \infty$ is a real number such that

$$\|x_n - Tx_n\| \leq M_3 \text{ and } \|Tx_n - w_n\| \leq M_3.$$

Substitute (2.9) into (2.8) we obtain:

$$\begin{aligned} \|x_{n+1}-x^*\| &\leq [1-\alpha_n(1-k_1(1-\beta_n))] \|x_n-x^*\| \\ &+ \alpha_n\beta n k_1[(1-\Upsilon_n(1-k_1)) \|x_n - x^*\| \\ &+\Upsilon_n(k_2+1)M_3]+\alpha_n\beta_n k_1(k_2+1)M_2 \\ &+\alpha_n(k_2+1)M_1 \quad (2.8) \\ &= \{[1-\alpha_n(1-k_1(1-\beta_n))] \\ &+\alpha_n\beta_n k_1(1-\Upsilon_n(1-k_1))\} \|x_n-x^*\| \\ &+ \alpha_n\beta_n\Upsilon_1 k_1(k_2+1)M_3 + \alpha_n\beta_n\Upsilon_1 k_1(k_2+1)M_2 \\ &+ \alpha_n(k_2+1)M_1 \\ &= \{1-\alpha_n+\alpha_n k_1- \alpha_n\beta_n k_1+ \alpha_n\beta_n k_1 \\ &\quad - \alpha_n\beta_n k_1\Upsilon_n(1-k_1)\} \|x_n-x^*\| \\ &+\alpha_n(k_2+1)[\beta_n\Upsilon_1 k_1+ \beta_n k_1+1]M_4 \\ &=\{1-\alpha_n+\alpha_n k_1- \alpha_n\beta_n k_1\Upsilon_n(1-k_1)\} \|x_n-x^*\| \\ &+\alpha_n(k_2+1)[\beta_n\Upsilon_1 k_1+ \beta_n k_1+1]M_4 \\ &= [1-\alpha_n(1-k_1)(1+\beta_n\Upsilon_1 k_1)] \|x_n-x^*\| \\ &+\alpha_n(k_2+1)[\beta_n\Upsilon_1 k_1+ \beta_n k_1+1]M_4 \quad (2.10) \end{aligned}$$

Let $t_n = \alpha_n(1-k_1)(1+\beta_n\Upsilon_1 k_1)$,
$\sigma_n = \alpha_n(k_2+1)[\beta_n\Upsilon_n k_1+\beta_n k_1+1]M_4$
And $\rho_n = \|x_n-x^*\|$. Then (2.10) reduces to
$\rho_{n+1} = (1-t_n)\rho_n + \sigma_n$
From our hypothesis, we observe that $t_n \geq 0$ and $k_1 \leq 1$ implies that:

$$t_n = \alpha_n(1-k_1)(1+\beta_n\Upsilon_1 k_1) \leq 1,$$

Thus $0 \leq t_n \leq 1$. Observe further that $\sum t_n = \infty$, since $\sum\alpha_n = \infty$ and $\sigma_n = o(t_n)$. Hence, by Lemma 2.1, $\rho_n \to 0$ as $n \to \infty$. i.e.,$\{x_n\}$ converges strongly to x^*. This completes the

Remark: Theorem 2.2 extends the results of Naimpally and Singh [2], Chidume [3, 4], Chidume and Osilike [7] Rhoades [8] and Osilike [9] to the generalized Ishikawa type iteration procedure and also to the more general quasi-contractive definition.
We now extend Theorem 2.2 above to the slightly more general iteration method (1.3) above in the following:

Theorem 2.3: Let B, K, T be as in Theorem 2.3 above and satisfying contractive definition (1.1). Define a sequence $\{x_n\}$ iteratively for arbitrary $x_1 \in$ Kby :

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n Ty_n + c_n Tx_n \\ y_n &= a'_n x_n + b'_n Tz_n + c'_n v_n \\ y_n &= a''_n x_n + b'''_n Tx_n + c''_n w_n \end{aligned}\right\} \quad n \geq 1$$

where $\{v_n\}$, $\{w_n\}$ are bounded sequences in K and $\{a_n\}$, $\{a'_n\}$, $\{a''_n\}$, $\{b_n\}$, $\{b'_n\}$, $\{b''_n\}$, $\{b''_n\}$,$\{c_n\}$,$\{c'_n\}$,$\{c''_n\}$, are real sequences in [0, 1] satisfying the following conditions:

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1$,

(ii) $\sum b_n = \infty$. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b'_n = \lim_{n\to\infty} b''_n = 0$

(iii) $\alpha_n := b_n + c_n, \beta_n := b'_n + c'_n, \Upsilon_n := b''_n + c''_n$

(iv) $\dfrac{s_1 - s_3}{1 - s_3} < 1$

Then $\{x_n\}$ converges strongly to the fixed point of T.

Proof: Let p be a fixed point of T. Then

$$\begin{aligned} \|x_{n+1}-p\| &= \|a_n x_n + b_n Ty_n + c_n Tx_n - p\| \\ &= \|(1-\alpha_n)(x_n-p)+\alpha_n(Ty_n-p) \\ &+c_n(Tx_n-Ty_n)\| \quad (2.11) \\ &\leq (1-\alpha_n)\|x_n-p\| +\alpha_n\|Ty_n-p\| \\ &+c_n\|Tx_n-Ty_n\| \end{aligned}$$

Since T satisfies the condition (1.1) above, we have:

$$\|Ty_n-p\| \leq(k_1 \|y_n-p\| \; k_2 \|y_n-Ty_n\| \quad (2.12)$$

where, k_1, k_2 are the same as in Theorem 2.2 and by continuity of T, there exists $M_6 < \infty$
such that $\|y_n-Ty_n\| \leq M_6$ and $\|Ty_n-Tx_n\| \leq M_6$.
Substituting into (2.12), yields

$$\begin{aligned} \|x_{n+1}-p\| &\leq (1-\alpha_n)\|(x_n-p\| +\alpha_n[k_1 \|y_n-p\| \\ &+k_2 M_6]+ \alpha_n M_6 \\ &= (1-\alpha_n \|(x_n-p\| +\alpha_n k_1 \|y_n-p\| \\ &+\alpha_n(k2+1)M_6 \quad (2.13) \end{aligned}$$

We also have

$$\begin{aligned} \|y_n-p\| &= \|a_n x_n + b_n Tz_n + c'_n u_n - p\| \\ &\leq \|(1-\beta_n)(x_n-p)+ \beta_n(Tz_n-p) \\ &+ c'_n (u_n-Tz_n)\| \quad (2.14) \\ &\leq (1-\beta_n) \|x_n-p\| +\beta_n \|Tz_n-p\| \\ &+ c'_n \|u_n-Tz_n\| \end{aligned}$$

Quasi-contractive condition on T implies that:

$$\|Tz_n-p\| \leq k_1 \|z_n-p\| + k_2 \|z_n-Tz_n\| \quad (2.15)$$

Continuity of T implies that there exists a real number M such that:

$u_n - Tz_n \leq M_7$ and $z_n - Tz_n \leq M_7$

Substituting (2.15) into (2.14), we obtain:

$$y_n - p \leq (1-\beta_n)\ x_n - p + \beta_n k_1\ z_n - p + \beta n(k_2+1)M_7 \quad (2.16)$$

We also have:

$$z_n - p \leq (1-\gamma_n)\ x_n - p + \gamma_n\ Tx_n - p + c_n''\ v_n - Tx_n$$

and

$$Tx_n - p \leq k_1\ x_n - p + k_2\ x_n - Tx_n$$

Furthermore, continuity of T on the and bounded set K implies that there exists $M_8 < \infty$ such that $v_n - Tx_n \leq M_8$ and $x_n - Tx_n \leq M_8$. So that

$$z_n - p \leq (1-\gamma_n)\ x_n - p + \gamma_n k_1\ x_n - p + \gamma_n(k_2+1)M_8 \quad (2.17)$$

Substitute (2.17) into (2.16), we obtain:

$$\begin{aligned} y_n - p \leq\ & (1-\beta_n)\ x_n - p + \\ & \beta_n k_1[1-\gamma_n(1-k_1)\ x_n - p \\ & + \beta_n k_1 \gamma_n (k_2+1)M_8 + \beta_n(k_2+1)M_7 \quad (2.18) \\ & [1-\beta_n(1-k_1(1-\Upsilon_n(1-k_1)))] \\ & + \beta_n(k_2+1)(k_1\Upsilon_n M_8 + M_7) \end{aligned}$$

Substituting (2.18) into (2.13) yields:

$$\begin{aligned} x_{n+1} - p \leq\ & (1-\alpha_n)\ x_n - p \\ & + \alpha_n k_1[1-\beta_n(1-k_1(1-\Upsilon_n(1-k_1)))]\ x_n - p \\ & + \alpha_n k_1\ \beta_n(k_2+1)(k_1\ \gamma_n M_8 + M_7) \\ & + \alpha_n(k_2+1)M_6 \quad (2.19) \\ = & [1-\alpha_n + \alpha_n k_1 - \alpha_n\beta_n k_1 \\ & + \alpha_n\beta_n k_1^2(1-\Upsilon n(1-k_1))]\ x_n - p \\ & + \alpha_n(k_2+1)\beta n\Upsilon n k_1 + \beta_n + 1]M_9 \end{aligned}$$

where, $M_9 = \max[M_6, M_7, M_8]$.

Since $k_1^2 < k_1$, we have $\alpha_n\beta_n k_1^2 < \alpha_n\beta_n k_1$.

Let $\sigma_n = \alpha_n(k_2+1)[\beta_n\Upsilon_n k_1 + \beta_n + 1]M_9$. Then (2.19) reduces to

$$\begin{aligned} x_{n+1} - p & \leq [1-\alpha_n + \alpha_n k_1 - \\ & \quad \alpha_n\beta_n\Upsilon_n k_1(1-k_1)]\ x_n - p + \sigma_n \\ & = [1-\alpha_n(1-k_1(1-\beta_n\Upsilon_n(1-k_1))] \quad (2.20) \\ & \quad x_n - p + \sigma_n \end{aligned}$$

Put

$t_n = \alpha_n(1-k_1(1-\beta_n\Upsilon_n(1-k_1)$ and $\rho_n = x_n - p$.

Then (2.20) becomes

$\rho_{n+1} = (1-t_n)\rho_n + \sigma_n$

Clearly, $0 \leq t_n \leq 1$ and $\sum t_n = \infty$. Also $\sigma_n = o(t_n)$.

Hence, by Lemma 2.1, $\rho_n \to 0$ as $n \to \infty$. This implies that $\{x_n\}$ converges strongly to p. The proof is complete.

Remark: Theorem 2.3 is an improvement on Theorem 2.2. Hence, it also generalizes the results of Chidume [4], Chidume and Osilike [7], Rhoades [8] and Osilike [9].

We now consider the general iteration method (1.2) above and establish its convergence to the fixed point of operator satisfying (1.1) in the following.

Theorem 2.4: Let K be a nonempty closed bounded convex subset of a Banach space B. Suppose T and S are uniformly continuous selfmappings of K and T satisfies the quasi-contractive definition (1.1). Define sequence iteratively for arbitrary by:

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n Ty_n + c_n Sx_n \\ y_n &= a_n' x_n + b_n' Sz_n + c_n' v_n \\ z_n &= a_n'' x_n + b_n''' Tx_n + c_n'' w_n \end{aligned}\right\} \quad n \geq 1$$

where, $\{v_n\}$, $\{w_n\}$ are arbitrary sequences in K and $\{a_n\}$, $\{a_n'\}$, $\{a_n''\}$, $\{b_n\}$, $\{b_n'\}$, $\{b_n''\}$, $\{b_n''\}$, $\{c_n\}$, $\{c_n'\}$, $\{c_n''\}$, are real sequences in [0,1] satisfying

(i) $a_n + b_n + c_n = a_n' + b_n' + c_n' = a_n'' + b_n'' + c_n'' = 1$,

(ii) $\sum b_n = \infty$. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b_n' = \lim_{n\to\infty} b_n'' = 0$

(iii) $\alpha_n := b_n + c_n\beta_n := a_n' + b_n' + c_n'\Upsilon_n := b_n'' + c_n'' +$

(iv) $k_1 = \dfrac{s_1 - s_3}{1 - s_3} < \dfrac{1}{2}$

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof: Let p be a fixed point of T. Then

$$\begin{aligned} x_{n+1} - p &= a_n x_n + b_n Ty_n + c_n Sx_n - p \\ &= (1-\alpha_n)(x_n - p) + \alpha_n(Ty_n - p) \\ &\quad + c_n(Sx_n - Ty_n) \\ &\leq (1-\alpha_n)\ x_n - p + \alpha_n\ Ty_n - p \\ &\quad + c_n\ Sx_n - Ty_n) \quad (2.21) \\ &\leq (1-\alpha_n)\ x_n - p + \alpha_n\ Ty_n - p \\ &\quad + c_n\ Sx_n - p\ Ty_n - p)\ ] \\ &= (1-\alpha_n)\ x_n - p + 2\alpha_n\ Ty_n - p \\ &\quad + \alpha_n\ Sx_n - Ty_n) \end{aligned}$$

Now, by quasi-contractive property of T, we have:

$$Ty_n - p \leq k_1\ y_n - p + k_2\ y_n - Ty_n$$

By continuity of T and S on K, there exists a real number $M_1 < \infty$ such that $y_n - Ty_n \leq M_1$ and $Sx_n - p \leq M_1$. Substitute into (2.21), we have:

x_{n+1}-p $\leq$ (1-α_n) x_n-p +2α_n(k_1 y_n-p + k_2M_1) + α_nM_1
$\leq$ (1-α_n) x_n-p +2α_nk_1 y_n-p + α_n(1+2k_2)M_1 (2.22)

From our hypothesis and applying similar procedure as above, we have:

y_n-p $\leq$(1-β_n) x_n-p +β_n Sz_n-p +β_n Sz_n-v_n (2.23)

since $c'_n \leq \beta_n$. Continuity of S on K implies that there exists a real number $M_2 < \infty$ such that: Sz_n-p $\leq M_2$ and Sz_n-v_n $\leq$. M_2. Substituting into (2.23), we have:

y_n-p $\leq$ (1- β_n) x_n-p + 2 β_n M_2 (2.24)

substituting (2.24) and (2.22) yields

x_{n+1}-p $\leq$ (1-n) x_n-p + 2 nk_1(1- n) x_n-p + 4 α_n β_n k_1M_2 + α_n (1+2k_2)M_1
$\leq$ [1-α_n(1-2k_1(1-β_n))] x_n - p +α_n(β_nk_1+2k_2+1)M (2.25)

where, M = max{M_1,M_2}. Now, put
t_n= α_n(1-2k_1(1- β_n)), and σ_n = α_n($\beta_n k_1$+2k_2 +1)M
and let ρ_n = x_n- p , then (2.25) becomes
ρ_{n+1} = (1- t_n) + σ_n
Clearly $t_n \in$ [0,1] since $k_1 \leq \frac{1}{2}$ and αn, $\beta_n \in$ [0,1]. Also, sigma$_n$ = o(t_n). Therefore, by Lemma 2.1, $\rho_n \to 0$, as $n\to\infty$. This implies that {x_n} converges strongly to p – the fixed point of T. The proof is complete.

Remark: Theorem 2.4 generalizes both Theorem 2.2 and Theorem 2.3 above, hence it is a generalization of the results of Rhoades [8], Chidume [3, 4], Chidume and Osilike [7] and Osilike [9].

Theorem 2.5: Let K be a nonempty closed bounded convex subset of an arbitrary real Banach space B. Suppose S and T are uniformly continuous quasi-contractive selfmappings of K satisfying (1.1). Define sequence {x_n}iteratively by (1.2) above, where, {a_n}, {a'_n}, {a''_n}, {b_n}, {b'_n}, {b''_n}, {b''_n},{c_n},{c'_n},{c''_n}, are real sequences in [0,1] satisfying the following conditions:

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1$,

(ii) $\sum b_n = \infty$. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b'_n = \lim_{n\to\infty} b''_n = 0$

(iii) $a_n := b_n + c_{n\beta n} := b'_n + c'_n \Upsilon_n := b''_n + c''_n$

(iv) $k_1 \leq \dfrac{1-\alpha_n}{\alpha_n\beta_n\gamma_n}$

(v) $k_1 = \dfrac{s_1 - s_3}{1-s_3} < \dfrac{1}{3}$

Suppose S, T have a common fixed point in K. Then {x_n} converges strongly to the common fixed point of S and T.

Proof: Let p be the common fixed point of S and T. Then

x_{n+1}-p = a_nx_n+b_nTy_n+ c_nSx_n-p
= (1-α_n)(x_n-p)+α_n (Ty_n-p) +c_n(Sx_n-Ty_n)
$\leq$ (1-α_n) x_n-p +α_n Ty_n-p +c_n Sx_n-Ty_n) (2.26)
$\leq$ (1-α_n) x_n-p +α_n Ty_n-p + c_n[Sx_n-p Ty_n-p)]
$\leq$ (1-α_n) x_n-p +2α_n Ty_n-p + α_n Sx_n-p)

By quasi-contractive property of T, S, we have:
Ty_n-p $\leq k_1$ y_n-p +k_2 y_n-Ty_n

and
Sx_n-p $\leq k_1$ x_n-p +k_2 x_n-Sx_n

By continuity of S and T on the bounded set K, there exists a real number M1 < ∞ such that y_n-Ty_n $\leq M_1$ and x_n-Sx_n $\leq M_1$. Substituting into (2.26), we have:

x_{n+1}-p $\leq$ (1-α_n) x_n-p +2α_nk_1 y_n-p +2$\alpha_nk_2M_1$ + α_nk_1 x_n-p + $\alpha_nk_2M_1$
=(1- α_n(1-k_1)) x_n-p +2α_nk_1 y_n-p + 3$\alpha_nk_2M_1$ (2.27)

From our hypothesis,
y_n-p $\leq$ (1- β_n) x_n-p + β_n Sz_n-p + c'$_a$ Sz_n-v_n

Since S satisfies (1.1), we have:
Sz_n-p $\leq$ k_1 z_n-p + k_2 z_n-Sz_n

Therefore,
y_n-p $\leq$ (1- βn) x_n-p + β_nk_1 z_n-p + β_nk_2 z_n-Sz_n + β_n Sz_n-v_n

Continuity of S,T on K implies that there exists a real number M2<$\infty\leq$ such that z_n-Sz_n $\leq M_2$ and Sz_n-v_n $\leq M_2$. Therefore:

$$y_n\text{-}p \quad \leq (1-\beta_n) \quad x_n\text{-}p \ + \beta_n k_1 \quad z_n\text{-}p \\ + (\beta_n(k_2+1)M_2 \qquad (2.28)$$

We also have the following estimates.

$$z_n\text{-}p \quad \leq (1-\gamma_n) \ x_n\text{-}p \ + \gamma_n \ \ Tx_n\text{-}p \\ + c''_n \ \ Tx_n\text{-}w_n \\ \leq (1-\gamma_n) \ x_n\text{-}p \ + \gamma_n [k_1 \ \ x_n\text{-}p \qquad (2.29) \\ + k_2 \ \ x_n\text{-}Tx_n \ ] + \gamma_n \ \ Tx_n\text{-}w_n \\ = [1-\gamma_n (1\text{-}k_1)] \ \ x_n\text{-}p \ + \gamma_n (k_2+1)M_3$$

where, $M_3<\infty$ is a real number such that:
$x_n\text{-}Tx_n \ \leq M_3$ and $Tx_n\text{-}w_n \ \leq M_3 9$.
Substitute (2.29) into (2.28), we obtain:

$$y_n\text{-}p \quad \leq \ (1-\beta_n) \quad x_n\text{-}p \\ + \beta_n k_1[[1-\gamma_n (1\text{-}k_1)] \quad x_n\text{-}p \\ \gamma_n k_2 M_3 + c_n'' M_3] + \beta_n(k_2+1)M_2 \\ = [1-\beta_n(1\text{-}k_1(1-\gamma_n(1\text{-}k_1)))] \quad x_n\text{-}p \\ + \beta_n \gamma_n k_1 k_2 M_3 \qquad (2.30) \\ + \beta_n \gamma_n k_1 M_3 + \beta_n(k_2+1)M_2$$

Now, let $M = \max[M_1, M_2, M_3]$ and then substitute (2.30) into (2.27). We have

$$x_{n+1}\text{-}p \quad \leq (1-\alpha_n(1\text{-}k1) \quad x_n\text{-}p \ + \\ 2\alpha_n k_1 \ \ [1-\beta n(1\text{-}k1(1-\gamma n(1\text{-}k1)))] \\ x_n\text{-}p \quad + 2\alpha_n \beta_n k1[\gamma_n k_1 k_2 + \\ \gamma_n k_1 + k_2 + 1]M + 3\alpha_n k_2 M \qquad (2.31) \\ = [1-\alpha_n(1\text{-}k1) + \\ 2\alpha_n k_1(1-\beta_n(1\text{-}k_1(1-\gamma_n (1\text{-}k1)))] \quad x_n\text{-}p \\ + 2\alpha_n \beta_n k1[\gamma_n k_1 k_2 + \gamma_n k_1 + k_2 + 1]M + 3\alpha_n k_2 M$$

Let
$\sigma_n = +2\alpha_n \beta_n k1[\gamma_n k_1 k_2 + \gamma_n k_1 + k_2 + 1]M + 3\alpha_n k_2 M$
and

$$t_n \ = \alpha_n(1\text{-}k_1)\text{-}2\alpha_n k_1(1-\beta_n(1\text{-}k_1(1-\gamma_n(1\text{-}k1)))) \\ = \alpha_n[(1\text{-}k_1)\text{-}2k_1(1-\beta_n(1\text{-}k_1(1-\gamma_n(1\text{-}k1))))] \\ = \alpha_n[(1\text{-}3k_1)\text{-}2k_1(1-\beta_n(1\text{-}k_1(1-\gamma_n(1\text{-}k1))))] \\ \geq 0$$

Since, $k_1 \leq \dfrac{1}{3}$

Observe that $k_1 \leq \dfrac{1}{3}$ implies that $k_1 \leq \dfrac{1}{2}$ and using the fact that $1\text{-}k_1 < 1$, we have:
$t_n \leq \alpha_n[1\text{-}2k_1(1-\beta_n(1\text{-}k_1(1-\gamma_n(1\text{-}k_1))))]$

Clearly,
$1-\beta_n(1\text{-}k_1(1-\gamma_n(1\text{-}k_1)))) \leq 1$

Therefore,
$2k_1(1-\beta_n(1\text{-}k_1(1-\gamma_n(1\text{-}k_1)))) \leq 1.$

Hence
$t_n \leq \alpha_n[1\text{-}2k_1(1-\beta_n(1\text{-}k_1(1-\gamma_n(1\text{-}k_1))))] < 1$

Thus $0 \leq t_n \leq 1$. Observe that since $\sum\alpha_n = \infty$, then $\sum t_n = \infty$ Also, $\sigma_n = o(t_n)$.
Put $\rho_n = \ x_n\text{-}p$. Then (2.31) reduces to
$\rho_{n+1} = (1\text{-}t_n)\rho_n + \sigma_n$

Hence, by Lemma 2.1, $\rho_n \to 0$, as $n \to \infty$. This implies that $\{x_n\}$ converges strongly to the common fixed point of S and T. The proof is complete.

Remark: Theorem 2.5 is an improvement on the result of Owojori and Imoru [11] which itself is an extension of the previous results of Rhoades [8], Qihou [5] and Osilike [9] to the generalized Ishikawa type iteration method and to the common fixed point of the operators involved.
We now consider the situation when the two nonlinear operators in the iteration scheme satisfy different contractive definitions. In particular, we investigate the common fixed point of S and T when T is a uniformly continuous quasi-contractive operator in the sense of (1.1) and S is a k-contractive operator, where $k \in (0,1)$, in an arbitrary real Banach space. Our result is the following.

Theorem 2.6: Let T, K, B and $\{x_n\}$ be as defined in Theorem 2.5 above. Suppose S: K→ K is k – contractive, $k \in (0,1)$ and the condition (iv) on the parameters in Theorem 2.5, is replaced with $k_1 \leq \frac{1}{2}(1\text{-}k)$ and all other conditions in Theorem 2.5 are satisfied. Suppose S, T have a common fixed point in K. Then, $\{x_n\}$ converges strongly to the common fixed point of S and T, if it exists.

Proof: Let p be the common fixed point of S and T. By similar procedure as in the proof of Theorem 2.5, we have the following estimates:

$$x_{n+1}\text{-}p \quad \leq (1-\alpha_n) \quad x_n\text{-}p \ + \alpha_n \ \ Ty_n\text{-}p \\ + c_n \ \ Ty_n\text{-}Sz_n \\ \leq (1-\alpha_n) \quad x_n\text{-}p \ + \alpha_n k_1 \ \ y_n\text{-}p \\ + \alpha_n k_2 \ \ y_n\text{-}Ty_n \\ + \alpha_n \ \ Ty_n\text{-}p \ + \alpha_n \ \ Sx_n\text{-}p \qquad (2.32) \\ \leq (1-\alpha_n) \quad x_n\text{-}p \ + 2\alpha_n k_1 \ \ y_n\text{-}p \\ + 2\alpha_n k_2 \ \ y_n\text{-}Ty_n \\ + \alpha_n k \ \ x_n\text{-}p \\ = (1-\alpha_n(1\text{-}k)) \quad x_n\text{-}p \ + 2\alpha_n k_1 \ \ y_n\text{-}p \\ + 2\alpha_n k_2 \ \ y_n\text{-}Ty_n$$

We also have the following estimates:

$$y_n\text{-}p \quad \leq (1-\beta_n) \quad x_n\text{-}p \quad + \beta_n \ \ Sz_n\text{-}p \\ + c'_n \ \ Sz_n\text{-}v_n \qquad (2.33) \\ \leq (1-\beta_n) \quad x_n\text{-}p \quad + \beta_n k \ \ z_n\text{-}p \\ + \beta_n \ \ Sz_n\text{-}v_n$$

(since S is k-contractive)

By similar method as above, we have:

$$z_n\text{-}p \le (1\text{-}\gamma_n)\ x_n\text{-}p\ + \gamma_n\ Tx_n\text{-}p\ \ c''_n\ \ Tx_n\text{-}w_n$$
$$\le (1\text{-}\gamma_n)\ x_n\text{-}p\ + k_1\gamma_n\ x_n\text{-}p$$
$$+ k_2\gamma_n\ x_n\text{-}Tx_n\ \ c''_n\ \ Tx_n\text{-}w_n \quad (2.34)$$
$$+(1\text{-}\gamma_n(1\text{-}k_1))\ x_n\text{-}p\ + k_2\gamma_n\ x_n\text{-}Tx_n$$
$$+ \gamma_n\ Tx_n\text{-}w_n$$

Substitute (2.34) into (2.33) and observe that the continuity of S,T on bounded set K implies that there exists a positive real number $N_1 < \infty$ such that:

$$x_n\text{-}Tx_n \le N_1,\quad Tx_n\text{-}w_n \le N_1 \text{ and } Sx_n\text{-}v_n \le N_1$$

we have:

$$x_n\text{-}p \le (1\text{-}\beta_n)\ x_n\text{-}p\ + \beta_n k(1\text{-}\gamma_n(1\text{-}k_1))\ x_n\text{-}p$$
$$+ [\beta_n k k_2\gamma_n + \beta_n k\gamma_n + \beta_n]N_1$$
$$\le [1\text{-}\beta_n + \beta_n k(1\text{-}\gamma_n(1\text{-}k_1))]\ x_n\text{-}p \quad (2.35)$$
$$+ [\beta_n\gamma_n k(k_2+1) + \beta_n]N_1$$

Substitute (2.35) into (2.32) and observe that continuity of T on K implies that there exists a positive real number $N_2 < \infty$ such that $yn\text{-}Ty_n \le N_2$, we have:

$$x_{n+1}\text{-}p \le \{[1\text{-}\alpha_n(1\text{-}k)]+2\alpha_n k_1[1\text{-}\beta_n$$
$$+\beta_n k(1\text{-}\gamma_n(1\text{-}k_1))]\}\ x_n\text{-}p$$
$$+ [2\alpha_n k_1\beta_n\gamma_n k(k_2+1)+\beta_n$$
$$+ 2\alpha_n k_1\beta_n]N_1+2\alpha_n k_2 N_2 \quad (2.36)$$
$$= [1\text{-}\alpha_n(1\text{-}k)$$
$$+ 2\alpha_n k_1(1\text{-}\beta_n(1\text{-}k(1\text{-}\gamma_n(1\text{-}k_1))))]\ x_n\text{-}p$$
$$+ [2\alpha_n k_1\beta_n\gamma_n k(k_2+1)$$
$$+ \beta_n+2\alpha_n k_1\beta_n]N_1+2\alpha_n k_2 N_2$$

Let

$\sigma_n = [2\alpha_n k_1\beta_n\gamma_n k(k_2+1)+\beta_n+2\alpha_n k_1\beta_n]N_1+2\alpha_n k_2 N_2$

and

$t_n = \alpha_n[(1\text{-}k(\text{-}2k_1(1\text{-}\beta_n(1\text{-}k(1\text{-}\gamma_n(1\text{-}k_1))))]$

Also, let $\rho_n = x_n\text{-}p$ then (2.36) reduces to

$\rho_{n+1} \le (1\text{-}t_n)\rho_n + \sigma_n$

Now, $0 < k < 1$ implies that $0 < 1 - k < 1$. Therefore,

$t_n \le \alpha_n[1\text{-}2k_1(1\text{-}\beta_n(1\text{-}k(1\text{-}\gamma_n(1\text{-}k_1))))]$

Observe that, $(1\text{-}\beta_n(1\text{-}k(1\text{-}\gamma_n(1\text{-}k_1))) \le 1$ and from hypothesis $k_1 \le \frac{1}{2}(1\text{-}k)$ implies that $k_1 \le \frac{1}{2}$.

Thus, $2k_1(1\text{-}\beta_n(1\text{-}k(1\text{-}\gamma_n(1\text{-}k_1)))) \le 1$

and

$1\text{-}2k_1(1\text{-}\beta_n(1\text{-}k(1\text{-}\gamma_n(1\text{-}k_1)))) \le 1$

But $\alpha_n \le 1$, therefore,

$t_n \le \alpha_n[1\text{-}2k_1(1\text{-}\beta_n(1\text{-}k(1\text{-}\gamma_n(1\text{-}k_1))))] \le 1$

We can also write

$t_n = \alpha_n[(1\text{-}k)\text{-}2k_1(1\text{-}\beta_n(1\text{-}k(1\text{-}\gamma_n(1\text{-}k_1))))]$
$= \alpha_n[(1\text{-}k\text{-}2k_1)+2k_1\beta_n(1\text{-}k)+2kk_1\beta_n\gamma_n(1\text{-}k_1)]$

So that $t_n \ge 0$. Thus $0 \le t_n \le 1$, for all $n \to \infty$. . It is also clear that $\sum t_n = \infty$ and $\sigma_n = o(t_n)$.
Therefore, by Lemma 2.1, $\rho_n\ \rho \to 0$ as $n \to \infty$. This implies that $\{x_n\}$ converges strongly to p- the common fixed point of S and T. This completes the proof.

Remark: a generalized Mann type iteration method is defined for arbitrary x_1 in K by

$$x_{n+1} = a_n x_n + b_n Tx_n + c_n Sx_n$$

where, S,T are nonlinear self-mappings of nonempty compact convex subset of an arbitrary Banach space Band $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are real sequences in [0,1] satisfying

(i) $a_n + b_n + c_n = 1$, (ii) $\sum b_n = \infty$.

We now investigate the common fixed points of S and T for the generalized Mann type iteration method. Our result is the following.

Theorem 2.7: Let S, T, K, X be as in Theorem 2.5. Define sequence $\{x_n\}$ for arbitrary x_1 in K by
$x_{n+1} = a_n x_n + b_n Tx_n + c_n Sx_n$

where, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are sequences in [0,1] satisfying

(i) $a_n + b_n + c_n = 1$,
(ii) $\sum b_n = \infty$,
(iii) $\alpha_n := b_n + c_n = 1$
(iv) (iv) $k_1 = \dfrac{s_1 - s_3}{1 - s_3} \le \dfrac{1}{3}$

Suppose S,T have a common fixed point in K. Then the sequence $\{x_n\}$ converges strongly to the common fixed point of T and S.

Proof: Let p be a common fixed point of S and T. Then, from the hypothesis, we have:

$$x_{n+1}\text{-}p = (a_n x_n + b_n Tx_n + c_n Sx_n\text{-}p$$
$$= (1\text{-}\alpha_n)\ x_n\text{-}p$$
$$+\alpha_n\ Tx_n\text{-}p\ + c_n\ Sx_n\text{-}Tx_n$$
$$\le (1\text{-}\alpha_n)\ x_n\text{-}p\ + \alpha_n\ Tx_n\text{-}p$$
$$+ c_n\ Sx_n\text{-}Tx_n$$
$$\le (1\text{-}\alpha_n)\ x_n\text{-}p\ + \alpha_n\ Tx_n\text{-}p$$
$$+ \alpha_n[\ Sx_n\text{-}p\ +\ Tx_n\text{-}p\ ] \quad (2.37)$$
$$\le (1\text{-}\alpha_n)\ x_n\text{-}p\ + 2\alpha_n\ Tx_n\text{-}p$$
$$+ \alpha_n\ Sx_n\text{-}p$$
$$\le (1\text{-}\alpha_n)\ x_n\text{-}p\ + 2\alpha_n\ Tx_n\text{-}p$$
$$+ 2\alpha_n k_2\ x_n\text{-}Tx_n\ + \alpha_n k_1\ x_n\text{-}p$$
$$+ \alpha_n k_2\ x_n\text{-}Sx_n$$
$$= [1\text{-}\alpha_n(1\text{-}3k_1)]\ x_n\text{-}p\ + 3\alpha_n k_2 N_3$$

where, s_1, s_2 and s_3 are the constants in the general quasi-contractive definition (1.1), with

$k_2 = \frac{s_1 - s_3}{1 - s_3} < 1$ and $N_3 < \infty$ is a real number such that

$x_n\text{-}Tx_n \leq N_3$ and $x_n\text{-}Sx_n \leq N_3$

(by uniform continuity of S and T on the bounded set K).

Now, let $t_n = \alpha_n(1\text{-}3k_1)$. Then from hypothesis, $0 < t_n < 1$ and $\sum t_n = \infty$. Let $\sigma_n = 3\alpha_n k_2 N_3$. Then $\sigma_n = o(t_n)$.

Put $\rho_n = x_n\text{-}p$. Then (2.14) becomes

$\rho_{n+1} = (1\text{-}t_n)\rho_n + \sigma_n$

Therefore, by Lemma 2.1, $\lim_{n\to\infty} \rho_n = 0$. This implies that $\{x_n\}$converges strongly to p.

The proof is complete.

Remark: Theorem 2.7 is an extension of the results of Ganguly and Bandyopadhyay [6], Rhoades [8] to the general quasi-contractive definition (1.1) and to the common fixed point of the two operators involved in the generalized Mann type iteration procedure.

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