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Well-Posedness of Second Order Uniformly Elliptic Boundary Control Systems with Additive or Multiplicative Uncertainty

Ada Cheng
Department of Science and Mathematics, Kettering University
Flint, Michigan, 48504. USA

Abstract: Boundary control systems arises naturally in many applications. Any modelling of real systems involve uncertainty. Sufficient conditions for well-posededness for additive or multiplicative uncertainty are presented through the use of matrix theory.

Keywords: infinite-dimensional systems, boundary control, uncertainty, input-output stability.

INTRODUCTION

Boundary control systems arises naturally in many applications such as robotics and acoustics. Modelling of any real systems require a certain amount of idealizing assumptions, consequently there are always some error in the mathematical model.

The input/output map of a boundary control system is said to be well posed if the mapping is well-defined and bounded. This concept was first unified by Salamon[8]. An ill-posed input/output map indicates that the measured outputs are not continuously dependent on the inputs. This would lead to difficulties in the practical implementation of any such control system. Often, however, ill-posedness of the control system indicates modelling errors. If the ideal boundary control system is close to being ill-posed, this would also hints in incorrect modelling errors.

In this paper we consider two types of uncertainty on a well-posed ideal boundary control system. We give sufficient conditions for both types of uncertainty.

In section 2, we give a brief introduction to boundary control and some relevance results. In section 3, we present our result through the use of matrix theory.

PRELIMINARIES

In this section we give some background definitions and results needed for the next section. For a more detailed introduction to boundary control system see for e.g. [1, 3, 6, 7, 8, 9].

Definition 1: A *boundary control system* is defined by

$$\left.\begin{array}{rll} \frac{d}{dt}z(t) & = Lz, & z(0)=z_0 \\ \Gamma z(t) & = u(t), & \\ y(t) & = Kz(t). & \end{array}\right\} \quad (1)$$

The operators $L \in \mathcal{L}(\mathcal{Z},\mathcal{H})$, $\Gamma \in \mathcal{L}(\mathcal{Z},\mathcal{U})$ and $K \in \mathcal{L}(\mathcal{Z},\mathcal{Y})$. The spaces $(\mathcal{Z},\|\cdot\|_{\mathcal{Z}}),(\mathcal{H},\|\cdot\|_{\mathcal{H}})$, $(\mathcal{U},\|\cdot\|_{\mathcal{U}}),(\mathcal{Y},\|\cdot\|_{\mathcal{Y}})$ are all Hilbert spaces and $\mathcal{Z}$ is a dense subspace of $\mathcal{H}$ with continuous, injective embedding $\iota_{\mathcal{Z}}$.

Throughout, we shall assume that the boundary control system (1) satisfies the following assumptions:

[A1] The operator Γ is onto, ker Γ is dense in $\mathcal{H}$ and there exists $\mu \in R$ such that $\ker(\mu I - L)\cap \ker\Gamma = 0$ and $\mu I - L$ is onto $\mathcal{H}$.

[A2] For any $z_0 \in \mathcal{Z}$ with $\Gamma z_0 = 0$ there exists a unique solution of (Γ, L) in $C^1[0,T;\mathcal{H}] \cap C[0,T;\mathcal{Z}]$ depending continuously on z_0.

Definition 2: The abstract elliptic problem $(L,\Gamma)_e$ corresponding to the boundary control system (L,Γ), as defined in (1), is

$$\left.\begin{array}{ll} Lz & = sz, \quad s \in \mathbb{C} \\ \Gamma z & = u. \end{array}\right\} \quad (2)$$

Th solution $z \in \mathcal{Z}$ will be denoted by $z(s)$.

Let Ω be an open set in $\Re^n$. A linear second order differential operator in Ω is defined by

$$L(x,D) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)D_{ij} + \sum_{i=1}^{n} c_i(x)D_i + d(x). \quad (3)$$

We assume that the coefficients are sufficiently smooth and that the operator L is uniformly elliptic in Ω. More precisely,

[H1a] (Smoothness Condition 1) The coefficients $a_{ij}(x)$ are bounded and uniformly continuous in $\bar{\Omega}$ and the remaining coefficients are bounded and measurable in Ω.

[H1b] (Uniform Ellipticity)Define the principal part of L by

Corresponding Author: Ada Cheng, Department of Science and Mathematics, Kettering University, 1700 West Third Avenue, Flint, Michgian, 48504-4898, U.S.A

$$L^0(x,D) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)D_{ij} = D^t A(x) D,$$

where $A(x)$ is an $n \times n$ positive definite matrix with components $a_{ij}(x)$. Then L is *uniformly elliptic in* Ω if there exists a positive constant c_L such that for all $x \in \Omega, \xi \in \Re^n$,

$$|L^0(x,\xi)| \geq c_L|\xi|^2. \quad (4)$$

The boundary operator Γ is defined by

$$\Gamma(x,D) = b_0(x) + \sum_{i=1}^{n} b_{1i}(x)D_i = b_0(x) + B_1'(x)D, \quad (5)$$

where $B_1'(x) = \big(b_{11}(x), \ldots, b_{1n}(x)\big)$ and $D' = (D_1, \ldots, D_n)$. We impose the following condition on the operator B.

[H2] (Smoothness Condition 2) The coefficients of B are real. Also, $b_0(x) \in C^2(\partial\Omega)$ and $b_{1i}(x) \in C^1(\partial\Omega)$, for $i = 1, \ldots, n$.

Estimates of the solution to a uniformly elliptic boundary value problem depend on regularity of the region Ω. In particular, we will assume that

[H3] Ω is bounded and uniformly regular of class C^2.[2]

In general, it is non-trivial to show that a region is uniformly regular of class C^m. For our work, we are concerned only with bounded sets Ω in $\Re^n$ and cylinders of the form $\Omega \times \Re$ in $\Re^{n+1}$. It can be shown that if Ω is bounded with sufficiently smooth boundary, then $\Omega \times \Re$ is also uniformly regular.

In addition to **[H1a], [H1b], [H2]** and **[H3]**, we assume, unless stated otherwise, that Ω, L and Γ also satisfy the following:

[H4] (root condition) Let $L^0(x,D)$ denote the principal part of $L(x,D)$. For every pair of linearly independent real vectors ξ and η, the polynomial $L^0(x, \xi + \tau\eta)$ in τ has an equal number of roots with positive and negative imaginary parts.

[H5] (complementing condition) Let $B^0(x,D)$ denote the principal part of $B(x,D)$. Let x be an arbitrary point on $\partial\Omega$ and n be the outward normal unit vector to $\partial\Omega$ at x. For each tangential vector $\xi \neq 0$ to $\partial\Omega$ at x, let $\hat{\tau}$ be the root of the polynomial $L^0(x, \xi + \tau n)$ with positive imaginary part. Then $\hat{\tau}$ is not a root of $B^0(x, \xi + \tau n)$.

Theorem 1: (Theorem 6.6[1]) *Let Ω, L, Γ define a boundary control system with $\mathcal{H} = L^2(\Omega)$ and $\mathcal{U} = H^{\frac{1}{2}}(\partial\Omega)$. Assume that* **[H1]-[H5]** *are satisfied. Then there exist a positive constant R such that for any $z \in H^2(\Omega)$, $u \in \mathcal{U}$ satisfying $\Gamma z = u$ on $\partial\Omega$ and any complex number s on the open right half plane $\mathbb{C}_{R^2} := \{s : \text{Re}\, s > R^2\}$, the following inequality holds:*

$$\begin{aligned} & |s|^{1/2}\|z\|_{H^1(\Omega)} + \|z\|_{H^2(\Omega)} \\ \leq\; & m\Big[\|(L-s)z\|_{L^2(\Omega)} + |s|^{1/2}\|u\|_{L^2(\Omega)} \\ & + \|u\|_{H^1(\Omega)}\Big], \end{aligned} \quad (6)$$

where m is a positive constant dependent only on L and Ω.

MAIN RESULTS

We are now in position to present our results. Throughout this section we shall assume that the idealized boundary control system takes the form in (1) and satisfies assumptions **A1,A2** and **H1-H5**.

The additive uncertainty consider here takes the form

$$\left.\begin{aligned} \tfrac{d}{dt}z(t) &= L_a z, \qquad z(0) = z_0 \\ \Gamma z(t) &= u(t), \\ y(t) &= Kz(t). \end{aligned}\right\} \quad (7)$$

where L_a and L only differs in the principal part of L in the following way:

$$L_a^0(x,D) = D^t(A(x) + \Delta(x))D.$$

Similarly, the multiplicative uncertainty takes the form

$$\left.\begin{aligned} \tfrac{d}{dt}z(t) &= L_m z, \qquad z(0) = z_0 \\ \Gamma z(t) &= u(t), \\ y(t) &= Kz(t). \end{aligned}\right\} \quad (8)$$

where L_m and L only differs in the principal part of L in the following way:

$$L_m^0(x,D) = D^t A(x)\Big(I_n + \Delta(x)\Big)D.$$

Theorem 2: *Let the idealized boundary control system (1) with L being a uniformly elliptic operator be well-posed. Let c_L be a constant satisfying Equation (4). If $\Delta(x)$ is symmetric, then the boundary control system (7) with Neumann or Robin boundary condition is also well-posed provided that $|\min_{x\in\Omega} S(\Delta(x))| < c_L$, where $S(\Delta(x))$ denotes the spectrum of $\Delta(x)$.* **Proof:** Since $\Delta(x)$ is symmetric, it is orthogonally diagonalizable. Thus there exists an invertible matrix $P(x)$ and a diagonal matrix $D(x)$ such that $\Delta = PDP^t$ and $P^tP = I$. Consequently, it is enough to show that the result hold for diagonal matrix $\Delta(x) = diag(d_{ii})$ since

$$\begin{aligned} \Delta = PDP^t \Rightarrow \xi^t\Delta\xi &= (P^t\xi)^t D(P^t\xi) \\ &\geq (\min_{x\in\Omega}\{d_{ii}\})|P^t\xi|^2 \\ &= (\min_{x\in\Omega}\{d_{ii}\})|\xi|^2 \end{aligned}$$

for all $\xi \in \Re^n$. To complete the proof, since L is uniformly elliptic there exists $c_L > 0$ such that $|\xi^t A\xi| \geq c_L|\xi|^2$. So for diagonal matrix $\Delta(x)$ we have that

$$|\xi^t(A+\Delta)\xi| \geq |\xi^t A\xi| - |\xi^t D\xi| \geq (c_L - |\min_{x\in\Omega}\{d_{ii}\}|)|\xi|^2.$$

Hence L_a is also uniformly elliptic and the result follows from Theorem 1.

Theorem 3: *Let the idealized boundary control system (1) with L being a uniformly elliptic operator be well-posed. Let c_L be a constant satisfying Equation (4). If $\Delta(x)$ is Hermitian, then the boundary control system (8) with Neumann or Robin boundary condition is also well-posed provided that $\min_{x\in\Omega}|S(\Delta(x))| < 1$.*

Proof: Let $\lambda_k(x)$, $k = 1, \ldots n$, denotes the eigenvalues of $\Delta(x)$. Since Δ is Hermitian, by Wely's Theorem on eigenvalues we have that $\lambda_k(\Delta) + 1 \leq \lambda_k(\Delta + I_n)$ for each $k = 1, \ldots, n$. Since $\min_{x\in\Omega}|S(\Delta(x))| < 1$, $\lambda_k(\Delta) + 1$ is positive for all k and so $I_n + \Delta(x)$ is positive definite. Since L is uniformly elliptic, $A(x)$ is positive definite thus so is $A(x)\big(I_n + \Delta(x)\big)$. Hence L_m is uniformly elliptic and by Theorem 1, system (8) is well-posed.

CONCLUSION

Well-posedness is preserved under additive and multiplicative uncertainty under mild assumption. These are sufficient but not necessary conditions since if $\Delta(x)$ is positive definite, then trivially the resulting boundary control system of the form (7) or (8) remains well-posed regardless of the bounds on its spectrum.

These mild assumption can be used to determine whether the idealized model is sufficiently accurate for use in controller design.

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