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Generalized Cauchy's Models and Generalized Integrals

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Abstract: The space of generalized complex numbers C^* has been constructed. The Cachy's model in the space of new generalized functions is well defined. The generalized integral of new generalized function over the compact *K* has been defined.

Key words: Cauchy's models, generalized complex numbers, generalized integral, ideal, algebra, topology

INTRODUCTION

Antonevich and Radyno[1] gave the following general method of constructing algebras of new generalized functions:

Let *E*- be some generalized function space and there is a some algebra *A* of infinitely many differentiable functions such that $A \subset E$.

The multiplication of generalized functions $\eta, \mu \in E$ will be defined by constructing a new algebra ζ and embedding (linear and injective mapping $j: E \to \zeta$, such that $j(uv)=j(u)j(v)$ for each $u, v \in A$).

If we have the following objects:

1. *E*- separated topological vector space;
2. Topological algebra $A \subset E$;
3. Some method of regularization define by a set of linear operators $R_{\psi,\varepsilon}: E \to A$, $\psi \in \phi$, $\varepsilon \in \varsigma$ (where ϕ- fixed set, ς- set with filter) so that $\forall\ \psi \in \phi, u \in E$

$R_{\psi,\varepsilon}(u) \to u$ in since of topology of *E*.

Define $G(\phi, A) = \{f : \phi \times \varsigma \to A\}$ and R_u the embedding of *E* into $G(\phi, A)$:

$E \ni u \to R_u(\phi, A)$, $R_u(\psi, \varepsilon) \equiv R_{\psi,\varepsilon}(u)$

The elements $f_1, f_2 \in G(\phi, A)$ are called weakly equivalent if $\forall \psi \in \phi$, $f_1(\psi,\varepsilon) - f_2(\psi,\varepsilon) \to 0$ in since of the topology of *E*.

In algebra $G(\phi, A)$ define a sub algebra $G^*(\phi, A)$ and some ideal $N(\phi, A)$ and define the algebra $\zeta(\phi, A) = G^*(\phi, A) / N(\phi, A)$.

Theorem 1: Let the sub algebra G^* and the ideal *N* satisfy the following conditions:

1. $\forall u \in E$, $R_u \in G^*$;
2. The elements of *N* are weakly equivalent of zero;
3. $R_{u.v} - R_u.R_v \in N$, $\forall\ u, v \in A$.

Then *E* included in algebra ζ as a vector sub space and *A* included in ζ as a sub algebra and if the operator of differentiation *D* defined in *A* so that $D(G^*) \subset G^*$ and $D(N) \subset N$ then the operator *D* is well defined on ζ and *A* embedded in ζ with the operator *D*.

Theorem 2: If there is an algebra ζ and embedding $j: E \to \zeta$, such that *A* included in ζ as a sub algebra. Then for each $R_{\psi,\varepsilon}$, $\psi \in \phi, \varepsilon \in \varsigma$, there are a sub algebras G^* and *N* that satisfy the conditions of Theorem 1 and $\zeta = G^* / N$ isometric of the smallest sub algebra containing *E*.

Generalized complex numbers: Following the Antonevich -Radyno general method of constructing algebras of new generalized functions in[2-7] were constructed many algebras of new generalized functions as: $\zeta(\xi(R))$, $\zeta(D(R))$, $\zeta(Z(R))$, $\zeta(S(R))$, $\zeta(\prod(R))$.

where the elements of the algebra $\zeta(M)$ are equivalence classes of sequences of elements in *M*.

To define the value of the element $\eta \in \zeta(M)$ at a some point x_0 and to define and study some mathematical models, for example as Cachy's problem:

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$$\begin{cases} Du = au \\ u(0) = c \end{cases}$$

We define the generalized complex numbers correspondence to the space of new generalized functions ζ by the following way:

Let *G(C)* - be the set of all sequences of complex numbers. Define $G^*(C)$ as the set of all sequences $(z_k) \in G(C)$ that is there are a natural number $j \in N$ and a constant $\sigma_1 > 0$, such that $|z_k| < \sigma_1 k^j$ for each *k* in the domain of sequence (z_k). Define the set $I^*(C)$ as the set of all sequences $(z_k) \in G(C)$ that is for each natural number $i \in N$, there is a constant $\sigma_2 > 0$, such that $|z_k| < \sigma_2 k^{-i}$ for each *k* in the domain of sequence (z_k).

Theorem 3

1. Each of sets *G(C)*, $G^*(C)$ is an algebra;
2. The set $I^*(C)$ be an ideal in the algebra $G^*(C)$.

Proof: We prove 2. Suppose that $\lambda = (\lambda_k)$ be an elements in $I^*(C)$ it implies that is for each a natural number $i \in N$, there is a constant $\sigma_1 > 0$, such that $|\lambda_k| < \sigma_1 k^{-i}$ for each k in the domain of sequence (λ_k) and let $\eta = (\eta_k) \in G^*(C)$ which implies that there are a natural number $j \in N$ and a constant $\sigma_2 > 0$, such that $|\eta_k| < \sigma_2 k^j$ for each *k* in the domain of sequence (η_k). The inequality $|\eta_k \lambda_k| \le |\eta_k|\,|\lambda_k| \le \sigma_2 k^j \sigma_1 k^{-i} = \sigma_2 \sigma_1 k^{j-i}$ implies that $\eta\lambda = (\eta_k \lambda_k) \in I^*(C)$.
The proof of 1 is similar.
Define the algebra of generalized complex numbers as a factor spaces
$C^* = G^*(C)/I^*(C)$.
The following theorem shows the importance of the construction of the algebra C^*:

Theorem 4

1. If $h = (h_k) \in G^*(\xi(R))$ and $\mu_0 \in R$, then $h(\mu_0) = (h_k(\mu_0)) \in G^*(C)$
2. If $\eta = (\eta_k) \in I^*(\xi(R))$ and $\mu_0 \in R$ then $\eta(\mu_0) = (\eta_k(\mu_0)) \in I^*(C)$.

Proof: It is not difficult to prove this theorem by using the definitions of the space $\zeta(\xi(R)) = G^*(\xi(R))/I^*(\xi(R))$ constructed in[2] and by the definition of the algebra of generalized complex numbers $C^* = G^*(C)/I^*(C)$ defined above.

Now we can define the value of the new generalized function $h \in \zeta(\xi(R))$ at each point $\mu_0 \in R$ as a generalized complex number $h(\mu_0) = (h_k(\mu_0)) \in C^*$, where (h_k) be any representative of the new generalized function $h \in \zeta(\xi(R))$.

We define the embeddings of the set of all real numbers *R* and the set of all complex numbers *C* into the space of complex generalized numbers $C^* = G^*(C)/I^*(C)$ by the following way:
$k_1 : x \in R \to (x_k + 0i) \in C^*$, where $x_k = x \ \forall k$.
$k_2 : z \in C \to (z_k) \in C^*$, where $z_k = z \ \forall k$.
The space $\zeta(\xi(R))$ together with the space C^* we will denote by $(\zeta(\xi(R)), C^*)$.
So the Cauchy's model in the space of new generalized functions is well defined in $(\zeta(\xi(R)), C^*)$ and has a general form:

$$\begin{cases} Du = vu \\ u(0) = z^* \\ u, v \in \zeta(\xi(R)),\ z^* \in C^* \end{cases}$$

Moreover there arise many mathematical models in the space $(\zeta(\xi(R)), C^*)$ which have mathematical sense. For example the following models

$$M_n = \begin{cases} Dv = \delta^n v \quad , \text{where } \delta - \text{the Dirac function} \\ u(a) = b \\ v \in \zeta(\xi(R)),\ a, b \in C^* \end{cases} \qquad n = 2,3,4,.....$$

has a mathematical sense in the algebra $(\zeta(\xi(R)), C^*)$.

Generalized integrals: We define the integral in the space $(\zeta(\xi(R)), C^*)$ as the following:
Let $K \subset R$ be any compact set and $\eta \in \zeta(\xi(R))$, define the integral of η over the compact K (which we denote by $\overrightarrow{\int_K} \eta(x)dx$) in the following way:

$$\overrightarrow{\int_K} \eta(x)dx = (\int_K \eta_k(x)dx),$$

where (η_k) – be any representative of η

Remark: The integral $\overset{\rightarrow}{\int_K} \eta(x)dx$ is well defined by virtue the following results:

Theorem 5

1. If $(\eta_k) \in G^*(\xi(R))$, then $(\int_K \eta_k(x)dx) \in G^*(C)$;
2. The integral $\overset{\rightarrow}{\int_K} \eta dx$ is independent on a representative (η_k);
3. If $(\lambda_k) \in I^*(\xi(R))$, then $(\int_K \lambda_k(x)dx) \in I^*(C)$.

Proof:

1. Since $(\eta_k) \in G^*(\xi(R))$, then there are $i \in N, d > 0$, such that $\sup_{x\in K}|\eta_k(x)| \le dk^i$ for each *k* in the domain of (η_k).

   Consider $\left|\int_K \eta_k(x)dx\right| \le \int_K |\ \eta_k(x)\ |dx \le dk^i \int_K dx$, that is $(\int_K \eta_k(x)dx) \in G^*(C)$.
2. Let $(\lambda_k - \lambda_k^/) \in I^*((\xi(R))$, then $\forall i \in N, \exists d > 0 : \sup_{x\in K}|\lambda_k(x) - \lambda_k^/(x)| \le dk^{-i}, \forall k$, consider

   $\left|(\int_K \lambda_k(x)dx - \int_K \lambda_k^/(x)dx\ )\right| =$

   $\left|(\int_K (\lambda_k(x) - \lambda_k^/(x))dx\ )\right| \le dk^{-i}\int_K dx\ \forall k$. Which means that $(\int_K \eta_k(x)dx - \int_K \eta_k^/\ (x)dx) \in I^*(C)$.
3. The proof of 3 is similar.

Definition: The generalized complex number z^* with representative $(\int_K \lambda_k(x)dx)$ is called the generalized integral of new generalized function $\lambda \in \zeta(\xi(R))$ over the compact *K*, that is:

$z^* = \overset{\rightarrow}{\int_K} \lambda(x)dx = (\int_K \lambda_k(x)dx)$, where $(\lambda_k(x))$ be any representative of λ.

The generalized integral defined above preserve many properties of usual integral defined in $\xi(R)$, for example the following properties are preserved:

1. $\overset{\rightarrow}{\int_K} [\lambda(x) \pm \eta(x)]dx = \overset{\rightarrow}{\int_K} \lambda(x)dx \pm \overset{\rightarrow}{\int_K} \eta(x)dx$;
2. $\overset{\rightarrow}{\int_K} a\lambda(x)dx = a\overset{\rightarrow}{\int_K} \lambda(x)dx$, $a \in C^*$.

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