

## Numerical Study on the Effect of Linear Friction of Long-Wave Propagation and Breaking with Sloping Bottom

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**Abstract:** In this study, we investigate the breaking of long-waves propagating on shallow water with linear friction on the sloping bottom. A complete set of equations is presented and a numerical method is developed to simulate the wave propagation. The method uses an up-wind difference scheme for the nonlinear convective term and the central difference scheme for other derivative terms. Various numerical examples have been conducted to investigate the effect of friction coefficient and drag coefficient on wave propagation and breaking.

**Keywords:** Friction, wave breaking, partial differential equations

### INTRODUCTION

Long-wave breaking on shallow water with a sloping bottom has been a matter of interest for most scientists over the last two centuries. Many attempts have been made to investigate and model such breaking. The first significant contribution was made by Carrier and Greenspan<sup>[3]</sup> who investigated the behaviour of a wave as it climbs up a sloping beach with no friction on the bottom. Okeke<sup>[14]</sup> investigated long-wave breaking on a sloping beach with governing linear differential equations. Kurkin and Pelinovsky<sup>[11]</sup> studied the mechanism of the spatial-temporal focusing of the edge waves in the self zone and proposed to find the localized anomalous high wave generating in the process of the wave packet focusing. Recently, Li and Jeng<sup>[12]</sup> used the conventional Korteweg-de Vries (KdV) equation to discuss the phenomena of nonlinear water wave propagation above the seabed with variable depth. However, friction on the bottom was not considered in the above papers, and consequently the results obtained do not agree well with the real wave propagation behaviour, particularly for the propagation of waves in the shallow water region. A very significant contribution on the study of the effect of bottom friction is due to Wu and Tian<sup>[26]</sup>, who studied the breaking of long waves propagating along an open channel with linear friction on the even bottom and derived an analytical solution for the estimation of breaking time and location. Based on Wu and Tian's work, this paper conducts an extensive numerical investigation to investigate the effect of bottom friction on wave

propagation and breaking on shallow water with sloping bottom.

### GOVERNING EQUATIONS

For the case where the undisturbed water depth is a linear function of the horizontal  $x$  coordinate as shown in Fig. 1.

The governing equations are as follows:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial}{\partial x} (\xi(x, t)) = - \frac{ku}{h(x) + \xi(x, t)},$$

$$\begin{aligned} & \frac{\partial}{\partial t} (h(x) + \xi(x, t)) + (h(x) + \xi(x, t)) \frac{\partial u}{\partial x}, \\ & + u \frac{\partial}{\partial x} (h(x) + \xi(x, t)) = 0 \end{aligned}$$

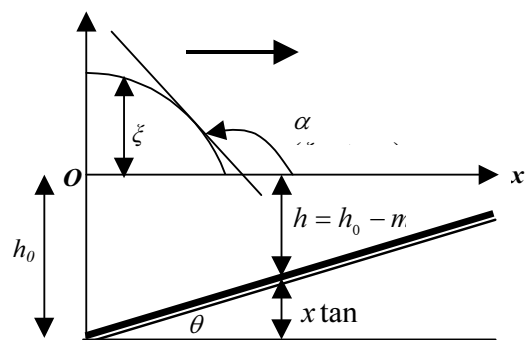


Fig. 1: Wave profile and coordinate system

where  $h(x) = h_0 - mx$ ,  $m = \tan \theta$ ,  $u = u(x, t)$  is the horizontal component of water velocity,  $h$  and  $\xi(x, t)$  represent respectively the undisturbed water depth and the water surface elevation,  $g$  denotes the gravitational acceleration,  $k$  is the drag coefficient,  $x$  and  $t$  are respectively horizontal coordinate and time.

**NUMERICAL SCHEME**

The governing equations for the problem in question can be written as:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial \xi(x, t)}{\partial x} + \frac{ku}{h(x) + \xi(x, t)} = 0, \tag{1}$$

$$\frac{\partial \xi(x, t)}{\partial t} + \frac{\partial}{\partial x} [(h(x) + \xi(x, t))u] = 0. \tag{2}$$

For the following time derivatives, we use the forward difference scheme,

$$\frac{\partial u}{\partial t} \approx \frac{u_{i,j+1} - u_{i,j}}{\Delta t}, \tag{3}$$

$$\frac{\partial \xi}{\partial t} \approx \frac{\xi_{i,j+1} - \xi_{i,j}}{\Delta t}. \tag{4}$$

For the following spatial derivatives, we use the central difference scheme,

$$\frac{\partial u}{\partial x} \approx \frac{u_{i+1} - u_{i-1}}{2\Delta x}, \tag{5}$$

$$\frac{\partial}{\partial x} [(h + \xi)u] \approx \frac{(h_{i+1} + \xi_{i+1})u_{i+1} - (h_{i-1} + \xi_{i-1})u_{i-1}}{2\Delta x}, \tag{6}$$

$$\frac{\partial}{\partial x} (\xi) \approx \frac{\xi_{i+1} - \xi_{i-1}}{2\Delta x}. \tag{7}$$

Thus by substituting (7) and (6) into (2), we obtain,

$$\xi_i^1 = \xi_i - \Delta t \left\{ \frac{(h_{i+1} + \xi_{i+1})u_{i+1} - (h_{i-1} + \xi_{i-1})u_{i-1}}{2\Delta x} \right\}. \tag{8}$$

For the nonlinear convective term  $u \frac{\partial u}{\partial x}$ , we use the upwind difference scheme. By using Taylor's theorem, the forward and backward difference formulae can be obtained respectively as follows:

$$\frac{du}{dx} \approx \frac{u_{i+1} - u_i}{\Delta x} = u^+, \tag{9a}$$

$$\frac{du}{dx} \approx \frac{u_i - u_{i-1}}{\Delta x} = u^-. \tag{9b}$$

The upwind difference scheme is defined by:

$$u \frac{\partial u}{\partial x} = uu^*, \tag{9c}$$

where

$$u^* = \begin{cases} u^- & ; \text{if } u > 0 \\ u^+ & ; \text{if } u < 0 \end{cases}$$

where  $u^+$  and  $u^-$  are defined by (9a) and (9b). For convenience in presentation, we introduce the sign function as below

$$\text{sgn}(u) = \frac{|u|}{u} \tag{9d}$$

Thus,

$$u_i^1 = u_i - \Delta t \left\{ \frac{1}{2\Delta x} \left[ \begin{aligned} &(u_i - |u_i|)(u_{i+1} - u_i) \\ &+ (u_i + |u_i|)(u_i - u_{i-1}) \\ &+ g(\xi_{i+1} - \xi_{i-1}) \end{aligned} \right] + \frac{ku_i}{h_i + \xi_i} \right\}. \tag{10}$$

Therefore, the finite difference scheme can be written as follows:

$$u_i^1 = u_i - \Delta t \left\{ \frac{1}{2\Delta x} \left[ \begin{aligned} &(u_i - |u_i|)(u_{i+1} - u_i) \\ &+ (u_i + |u_i|)(u_i - u_{i-1}) \\ &+ g(\xi_{i+1} - \xi_{i-1}) \end{aligned} \right] + \frac{ku_i}{h_i + \xi_i} \right\}, \tag{11a}$$

$$\xi_i^1 = \xi_i - \Delta t \left\{ \frac{(h_{i+1} + \xi_{i+1})u_{i+1} - (h_{i-1} + \xi_{i-1})u_{i-1}}{2\Delta x} \right\}, \tag{11b}$$

where all of the variables are as defined in section 2.

**NUMERICAL RESULTS**

Letting  $h_0 = 10$  meters, we have  $h(x) = 10 - mx$  where  $m = \tan \theta$  and the initial water level disturbance is chosen as<sup>[26]</sup>:

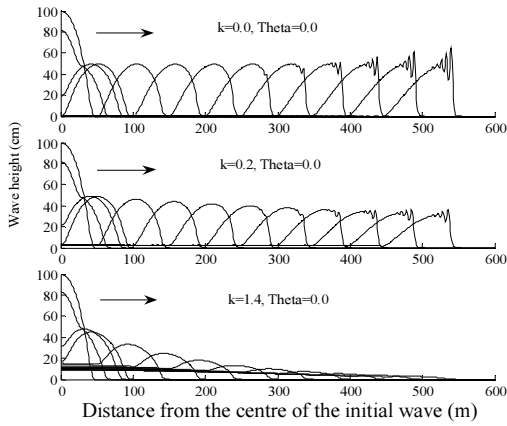


Fig. 2: Numerical results of evolution of long-waves for  $\theta = 0.0$  with different drag coefficients

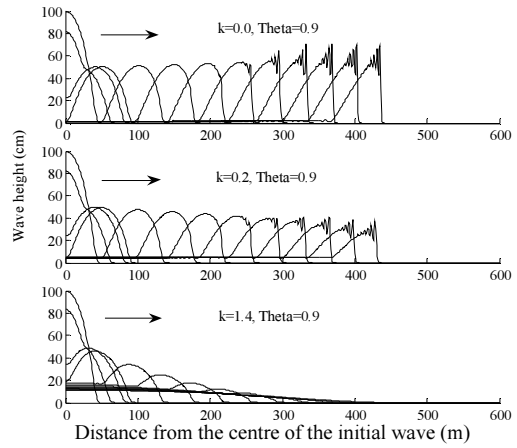


Fig. 5: Numerical results of evolution of long-waves for  $\theta = 0.9$  with different drag coefficients

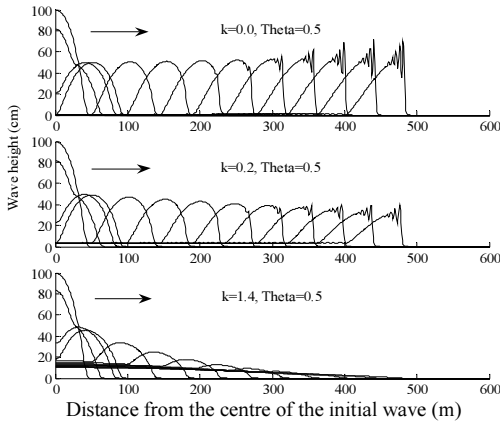


Fig. 3: Numerical results of evolution of long-waves for  $\theta = 0.5$  with different drag coefficients

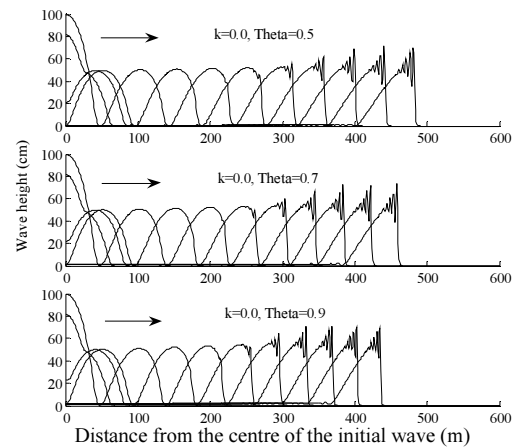


Fig. 6: Numerical results of evolution of long-waves for  $k = 0.0$  m/s with different  $\theta$

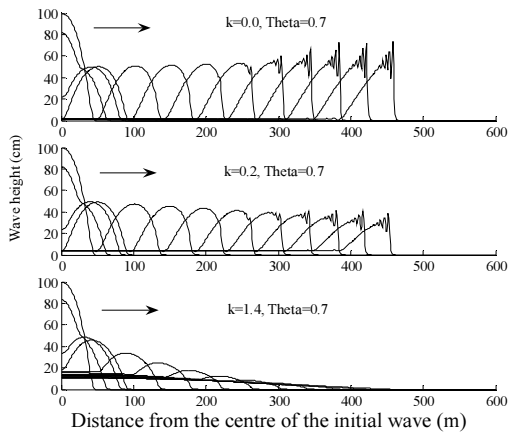


Fig. 4: Numerical results of evolution of long-waves for  $\theta = 0.7$  with different drag coefficients

$$\xi(x,0) = \begin{cases} \exp\left(-\frac{x^2}{50^2 - x^2}\right) & ; 0 < |x| < 50 \\ 0 & ; |x| \geq 50. \end{cases} \quad (12)$$

The numerical results for the problem considered here are as follows. The vertical axis and the horizontal axis represent, respectively, wave height (cm) and distance from the centre of the initial wave (m) respectively.

Fig. 2-5 show the effect of linear friction on long-wave propagation and breaking with different slope on the sloping bottom. These results also agree with the results obtained by Wu and Tian for the case with even bottom. Fig. 6-8 show the influence of slope on the long

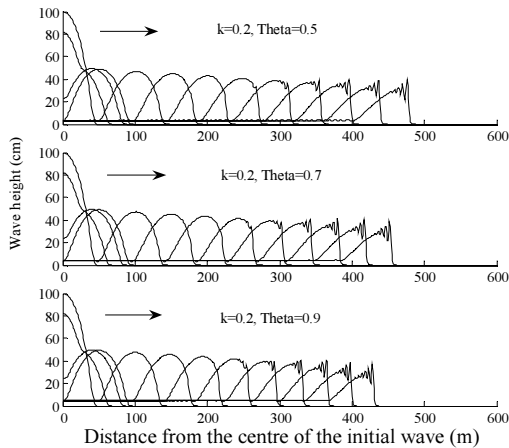


Fig. 7: Numerical results of evolution of long-waves for  $k = 0.2$  m/s with different  $\theta$

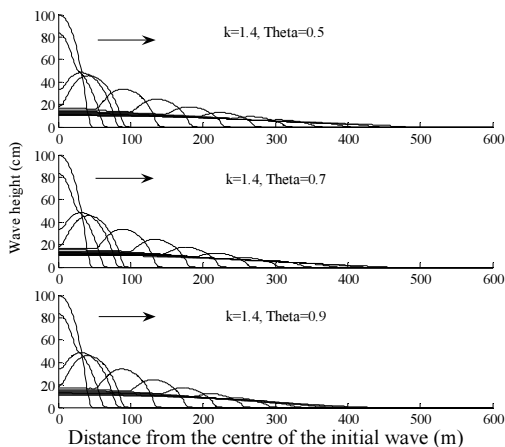


Fig. 8: Numerical results of evolution of long-waves for  $k = 1.4$  m/s with different  $\theta$

wave propagation and breaking. As noted from the Figs., if the slope of the bottom increases, the distance from the centre of the initial wave to the location where wave breaking occurs decreases. This is in agreement with reality: the slope of beaches affects the distance from the source to the beach.

### CONCLUSION

We have successfully simulated the effect of the linear friction on the evolution of long wave propagation and breaking for the case with sloping bottom. If the drag coefficient is less than the critical value, long wave breaking will take place. However, if the drag coefficient is greater than the critical value,

long-wave breaking will not occur. This is because for the large drag coefficient, the diffusion due to the bottom friction is predominant and the motion energy of the long-waves is gradually exhausted. It has also been found that with the increase in the slope of the bottom, the breaking time and the distance from the initial wave to the location where wave breaking takes place decrease.

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