

A Pseudo Spline Methods for Solving an Initial Value Problem of Ordinary Differential Equation

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Abstract: New scheme for solving initial value problem of ordinary differential equation was derived. Starting from the general method of deriving the spline function, the scheme was developed based on interpolation and collocation.

Key words: Initial value problems, ordinary differential equations, collocation, pseudo spline function

INTRODUCTION

Consider the Initial Value Problem (IVP):

$$\begin{aligned}y'(t) &= f(t, y(t)) \\ y(t_0) &= y_0\end{aligned}\quad (1)$$

where, $a \leq t \leq b$, $a = t_0 < t_1 < t_2 < \dots < t_{N-1} = b$, $N = \frac{(b-a)}{h}$, $N = 0, 1, \dots, N-1$ and $h = t_{n+1}$ is called the step length. The conditions on the function $f(t, y(t))$ are such that existence and uniqueness of solution is guaranteed^[2].

The numerical solution of the Eq. 1 had received lots of attention and it is still receiving such due to the fact that many physical (Engineering, Medical, financial, population dynamics and Biological sciences) problems formulated into mathematical equation results into the above type.

The solution is generated in a step-by-step fashion by a formula which is regarded as discrete replacement of the Eq. 1^[1-5].

In the class of methods available in solving the problem numerically, the most celebrated methods are the single-step and the multisteps methods. In a single step, an information at just one point is enough to advance the solution to the next point while for the multisteps (as the name suggests), information at more than one previous points will be required to advance the solution to the next point.

The Euler's method (the pioneering method), which is the oldest method and the Runge-Kutta methods fall in the class of the single-step methods while the Adams methods are in the class of the multistep method^[1-5]. The Adams method is divided into two namely the Adams-

Bashforth (explicit) and Adams-Moulton (implicit). These two methods combined can be used as a predictor-corrector method. This class of method has been proved to be one of the most efficient method to solve certain class of IVP (non-stiff).

In the literature, the derivation of the Adams method had been extensively dealt with using the interpolatory polynomial for the discretised problem. For the derivation of linear multisteps method through interpolation and collocation^[1,3,4,6], Omolehin *et al.*^[8] used the collocation method to derive a new class of the Adams-Bashforth schemes for ODE while Onumayi *et al.*^[9] also used the collocation method for deriving a continuous multisteps method. Lie *et al.*^[7] discussed the super convergence properties of the collocation methods.

In this study, we consider a new class of methods for solving (1) based on interpolation and collocation. Our method is based on a general method for deriving the spline functions.

The study is organised in the following order, §2 deals with description of piecewise interpolation functions, the derivation of our scheme features is §3. The results from some numerical examples will be given to illustrate and validate our scheme in §4. Furthermore, our results will be compared with an already known scheme of Omolehin *et al.*^[8] while the conclusion is featured in the last Section.

Piecewise-interpolation: One of the methods of deriving the multisteps method is by polynomial interpolation for a set of discrete point, however, polynomial interpolation for a set of $(N+1)$ points $\{t_k, y_k\}$ is frequently unsatisfactory because the interpolation error is related to higher derivatives of the interpolated function. To circumvent this, we discretise the interpolation domain and interpolate

locally. The overall accuracy may be significantly improved even if the interpolation polynomial is of low order.

Interpolation functions obtained on this principle are piece-wise interpolation functions or splines. We define a spline function as follows:

Definition: A function $S(t)$ is called a spline of degree k if:

- The domain of S is the interval $[a, b]$
- $S, S', S'', \dots, S^{(k-1)}$ are all continuous on $[a, b]$
- There are points t_i (called knots) such that $a = t_1 < t_2 < \dots < t_n = b$ and such that S is a polynomial of degree k on each sub-interval $[t_i, t_{i+1}]$, $i = 1, \dots, n - 1$. subject to the interpolating conditions
- $S(t_i) = y(t_i) \forall t \in [t_i, t_{i+1}]$ $i = 1, \dots, n-1$
- $S_r^{(j)}(t_i) = S_{r+1}^{(j)}(t_i)$, $j = 1, \dots, k-1$, $r = 1, n-1$, $i = 2, \dots, n-1$

Condition (iv) is the collocation while (v) is the continuity condition, only on interior knots.

We shall now use the piece-wise linear and cubic interpolation spline functions to derive our methods. Adams methods are recoverable from our methods

Derivation of the scheme:

Pseudo Quadratic spline function: Let $S(t)$ be the desired function, the linear langrange interpolation formula gives the following representation for $S'(t)$ at the given points t_{n-1} and t_n , for all $t \in [t_{n-1}, t_n]$, as:

$$\frac{S'(t) - S'(t_{n-1})}{(t - t_{n-1})} = \frac{S'(t_n) - S'(t)}{(t_n - t)} \tag{2}$$

Simplifying (2) we have:

$$S'(t) = \frac{1}{(t_n - t_{n-1})} \{ (t - t_{n-1}) S'(t_n) + (t_n - t) S'(t_{n-1}) \} \tag{3}$$

Integrating (3):

$$S(t) = \frac{1}{2(t_n - t_{n-1})} \left\{ \frac{1}{2} (t - t_{n-1})^2 S'(t_n) - \frac{1}{2} (t_n - t)^2 S'(t_{n-1}) + A \right\} \tag{4}$$

where, A is the constant of integration to be determined. Since $S(t)$ interpolates the function f at $t = t_n$, it implies that $S(t_n) = f(t_n, y(t_n))$.

Thus for $t = t_{n-1}$:

$$A = S(t_{n-1}) + \frac{1}{2(t_n - t_{n-1})} (t_n - t_{n-1})^2 S'(t_{n-1}) \tag{5}$$

Substitute (5) into (4) we have,

$$S(t) = S(t_{n-1}) + \frac{1}{2(t_n - t_{n-1})} (t_n - t_{n-1})^2 S'(t_{n-1}) + \frac{1}{2(t_n - t_{n-1})} \left\{ \frac{1}{2} (t - t_{n-1})^2 S'(t_n) - \frac{1}{2} (t_n - t)^2 S'(t_{n-1}) \right\} \tag{6}$$

$$S(t) = S(t_{n-1}) + \frac{S'(t_{n-1})}{2(t_n - t_{n-1})} \{ (t_n - t_{n-1})^2 - (t_n - t)^2 \} + \frac{S'(t_n)}{2(t_n - t_{n-1})} (t - t_{n-1})^2 \tag{7}$$

If in (4) we evaluate $S(t)$ at $t = t_n$:

$$A = S(t_n) - \frac{1}{2(t_n - t_{n-1})} (t_n - t_{n-1})^2 S'(t_n) \tag{5'}$$

If (5') is substituted into (4) we have:

$$S(t) = S(t_n) - \frac{S'(t_n)}{2(t_n - t_{n-1})} \{ (t_n - t_{n-1})^2 - (t - t_{n-1})^2 \} - \frac{S'(t_{n-1})}{2(t_n - t_{n-1})} (t_n - t)^2 \tag{8}$$

Collocating (7) and (8) at $t = t_{n+1}$ and using the property that $S(t) \approx y(t)$ and that $h = t_n - t_{n-1}$ we have the:

$$y_{n+1} = y_{n-1} + 2hf_n \tag{9}$$

and

$$y_{n+1} = y_n + \frac{h}{2} \{ 3f_n - f_{n-1} \} \tag{10}$$

If we also collocate (7) at $t = t_n$ and simplify we have:

$$y_n = y_{n-1} + \frac{h}{2} \{ f_n + f_{n-1} \} \tag{11}$$

Equation (9) and (10) correspond to the mid-point rule and the Adams-Bashforth of second order while (11) is an implicit method (the Implicit Trapezoidal Method).

Various multisteps of the Adams forms can be derived from the Eq. (7) and (8) at different collocation points (say $t = t_{n+2}, t_{n+3}, \dots$).

The local truncation error: Assume that $y \in C^3 [a, b]$ for all x in $a \leq x \leq b$. Due to a standard approach by Lambert^[5] we have been able to show that the local truncation errors associated with these numerical algorithms can be expressed respectively as:

$$e_9 = \frac{1}{3}h^2y'''(\zeta), \quad \zeta \in (x_{n-1}, x_{n+1})$$

$$e_{10} = \frac{5}{12}h^2y'''(\zeta), \quad \zeta \in (x_{n-1}, x_{n+1})$$

$$e_{11} = \frac{1}{12}h^2y'''(\zeta), \quad \zeta \in (x_{n-1}, x_n)$$

Using well known analysis in Herinci^[2] and Lambert^[5], it can be shown that these methods are all consistent and zero stable. Consistency and zero stability are necessary and sufficient conditions for the convergence of methods of this kind, hence the three numerical schemes are convergent with errors of order $O(h^2)$.

Pseudo cubic spline function: Since we are considering a piecewise cubic spline, its second derivative is piecewise linear on $[t_{n-1}, t_n]$, then the linear langrange interpolation formula gives the representation for $S''(t)$ at the given points t_{n-1} and t_n :

$$\frac{S''(t) - S''(t_{n-1})}{(t - t_{n-1})} = \frac{S''(t_n) - S''(t_{n-1})}{(t_n - t)} \quad (12)$$

$$S''(t) = \frac{1}{(t_n - t_{n-1})} \{ (t - t_{n-1})S''(t_n) + (t_n - t)S''(t_{n-1}) \} \quad (13)$$

Integrating Eq. (13) twice we have,

$$S(t) = \frac{1}{(t_n - t_{n-1})} \left\{ \frac{S''(t_n)}{6}(t - t_{n-1})^3 + \frac{S''(t_{n-1})}{6}(t_n - t)^3 \right\} + A(t_n - t) + B(t - t_{n-1}) \quad (14)$$

where, A and B are constants. To determine these constants, (14) is collocated at two points say $t = t_{n-1}$ and $t = t_n$, this yield:

$$S(t_{n-1}) = \frac{S''(t_{n-1})}{6}(t - t_{n-1})^2 + A(t_n - t_{n-1}) \quad (15)$$

and

$$S(t_n) = \frac{S''(t_n)}{6}(t - t_{n-1})^2 + B(t_n - t_{n-1}) \quad (16)$$

From (15) and (16) we have that:

$$A = \frac{1}{(t_n - t_{n-1})} \left\{ S(t_{n-1}) - \frac{S''(t_{n-1})}{6}(t_n - t_{n-1})^2 \right\}$$

and

$$B = \frac{1}{(t_n - t_{n-1})} \left\{ S(t_n) - \frac{S''(t_n)}{6}(t_n - t_{n-1})^2 \right\}$$

Substitute for A and B in (14), we have:

$$S(t) = \frac{1}{(t_n - t_{n-1})} \left\{ \frac{S''(t_n)}{6}(t - t_{n-1})^3 + \frac{S''(t_{n-1})}{6}(t_n - t)^3 \right\} + \frac{1}{(t_n - t_{n-1})} \left\{ S(t_{n-1}) - \frac{S''(t_{n-1})}{6}(t_n - t_{n-1})^2 \right\} (t_n - t) \quad (17) + \frac{1}{(t_n - t_{n-1})} \left\{ S(t_n) - \frac{S''(t_n)}{6}(t - t_{n-1})^2 \right\} (t - t_{n-1})$$

Collocating (17) at $t = t_{n+1}$ yields:

$$S(t_{n+1}) = 2S(t_n) - S(t_{n-1}) + h^2S''(t_n) \quad (18)$$

By collocation property, we have:

$$y_{n+1} = 2y_n - y_{n-1} + h^2y''_n \quad (19)$$

and using (1), we have that the coefficient of h^2 in the Eq. 19 can be replaced by:

$$y'' = f_t(t_n, y(t_n)) + f_y f_t(t_n, y(t_n)) \quad (20)$$

where, here f_t and f_y are the first partial derivatives of $f(t, y(t))$ with respect to t and y respectively. Using the approximation relations, $f_t \approx \frac{f_{n+1} - f_{n-1}}{2h}$ and

$f_y \approx \frac{f_{n+1} - f_{n-1}}{2h}$ we simplify (18) to give:

$$y_{n+1} = 2y_n - y_{n-1} + \frac{h}{2} \{ (1 + f_y)(f_{n+1} - f_{n-1}) \} \quad (21) + \frac{h}{2} \{ (f_{n+1} - f_{n-1}) + f_y f_{n+1} - f_y f_{n-1} \}$$

Neglecting the nonlinear part in (21), Eq. 21 becomes:

$$y_{n+1} = 2y_n - y_n + \frac{h}{2} \{f_{n+1} - f_{n-1}\} \tag{22}$$

which is an implicit 2-step method.

The local truncation error associated with (22) as outlined for the schemes (9)-(11) can be shown to be $-\frac{1}{12}h^3$. The scheme was observed to be consistent but to our surprise the method is not zero stable according to^[2,5] yet it gives a convergent solution of maximum error of order $O(h^3)$.

RESULTS

The schemes derived in this study are:

$$y_{n+1} = y_{n-1} + 2hf_n$$

$$y_{n+1} = y_n + \frac{h}{2} \{3f_n - f_{n-1}\}$$

$$y_n = y_{n-1} + \frac{h}{2} \{f_n + f_{n-1}\}$$

$$y_{n+1} = 2y_n - y_n + \frac{h}{2} \{f_{n+1} - f_{n-1}\}$$

Numerical examples: We shall consider the following problems:

1. $y' = -\frac{y}{2(t+1)}, y(0) = 1, t \in [0, 1]$

The exact solution is given as:

2. $y(t) = \frac{1}{\sqrt{1+t}}$
 $y' = y - t^2 + 1, y(0) = 0.5, t \in [0, 1]$

The exact solution is given as:

$$y(t) = (1+t)^2 - 0.5 * \exp(t)$$

DISCUSSION

The methods described by Eq. 9 and 10 are respectively represented as method A and B, while method C, D, E and F are the combinations of Eq. 9 with (11), (10) with (11), (9) with (22) and (10) with (22) as predictor-corrector methods respectively.

Table 1 and 2 shows the maximum error of the Methods A and B for the examples with $h = 0.1$.

Table 1: Error of y(t) for example 1 (h = 0.1)

t	Method A	Method B
0.3	1.8428727e-004	2.2810119e-004
0.4	2.4052639e-004	5.1440716e-004
0.5	4.5748153e-004	8.5876660e-004
0.6	5.8078449e-004	1.2697656e-003
0.7	8.4856274e-004	1.7575791e-003
0.8	1.0543354e-003	2.3338030e-003
0.9	1.3952231e-003	3.0116219e-003
1.0	1.7044890e-003	3.8060145e-003

Table 2: Error of y(t) for example 2 (h = 0.1)

t	Method A	Method B
0.3	4.5065362e-004	5.8284717e-004
0.4	2.9443804e-004	9.7450032e-004
0.5	6.7867750e-004	1.2528296e-003
0.6	4.3923840e-004	1.4525075e-003
0.7	8.0112923e-004	1.5966312e-003
0.8	5.1017146e-004	1.7007588e-003
0.9	8.6895411e-004	1.7756254e-003
1.0	5.4196872e-004	1.8287918e-003

Table 3: Error of y(t) for example 1 (h = 0.1) for methods A and B^[8]

t	Method A	Method B	[8]
0.4	2.9443804e-004	9.7450032e-004	4.2530000e-004
0.5	6.7867750e-004	1.2528296e-003	7.1200000e-004
0.6	4.3923840e-004	1.4525075e-003	1.0556000e-003
0.7	8.0112923e-004	1.5966312e-003	1.4620000e-003
1.0	5.4196872e-004	1.8287918e-003	3.1397000e-003

Table 4: Error of y(t) for example 1 (h = 0.1)

t	Method C	Method D	Method E	Method F
0.3	3.9207500e-005	3.7016804e-005	2.6716341e-005	2.3962309e-005
0.4	8.4890939e-005	8.1906006e-005	7.1444753e-005	6.3614004e-005
0.5	1.3983292e-004	1.3583079e-004	1.2883332e-004	1.1388709e-004
0.6	2.0539203e-004	2.0019529e-004	1.9547040e-004	1.7156573e-004
0.7	2.8319791e-004	2.7659929e-004	2.6911307e-004	2.3454770e-004
0.8	3.7510163e-004	3.6686362e-004	3.4824665e-004	3.0142337e-004
0.9	4.8320835e-004	4.7305897e-004	4.3182385e-004	3.7122645e-004
1.0	6.0991014e-004	5.9753835e-004	5.1910500e-004	4.4328173e-004

Table 5: Error of y(t) for example 2 (h = 0.1)

t	Method C	Method D	Method E	Method F
0.3	1.0547721e-004	1.0823124e-004	4.6064141e-006	6.7971098e-006
0.4	1.8156813e-004	1.8399233e-004	1.4349767e-005	2.1497021e-005
0.5	2.3569468e-004	2.3785106e-004	2.9862905e-005	4.5378172e-005
0.6	2.7460440e-004	2.7656997e-004	5.1892600e-005	7.9915113e-005
0.7	3.0275940e-004	3.0458348e-004	8.1313344e-005	1.2680491e-004
0.8	3.2316603e-004	3.2488108e-004	1.1914183e-004	1.8799699e-004
0.9	3.3789855e-004	3.3952672e-004	1.6655518e-004	2.6572679e-004
1.0	3.4841822e-004	3.4997514e-004	2.2491025e-004	3.6255372e-004

The numerical solution generated by these methods are compared with the third order method of^[8] and this is shown in Table 3.

From the tables of results displayed it could be seen that one of our methods which is of order 2 performs better than the third order method of^[8].

When the explicit methods of this work are combined to form a predictor-corrector method, the results as shown in Table 4 and 5 reveal that these methods give a better accuracy.

CONCLUSION

In this study, we have derived new schemes for solving first order differential equations based on interpolation and collocation through the general method for deriving the spline functions.

Although we pointed out in §3 that there are other explicit methods that could be obtained from collocating Eq. 7 and 8 from other points other than the ones used in this work, nevertheless these methods are far less accurate due to instability.

As a comparison to^[8], Eq. 7 and 8 are the continuous form of our scheme of order two with equivalent discrete forms (9) and (10). Our method is easier to derive and more user friendly than the method of derivation in^[8].

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