

Quadratic Interpolation Algorithm for Minimizing Tabulated Function

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Abstract: Problem statement: The problem of finding the minimum value of objective function, when we know only some values of it, is needed in more practical fields. Quadratic interpolation algorithms are the famous tools deal with this kind of these problems. These algorithms interested with the polynomial space in which the objective function is approximated. **Approach:** In this study we approximated the objective function by a one dimensional quadratic polynomial. This approach saved the time and the effort to get the best point at which the objective is minimized. **Results:** The quadratic polynomial in each one of the steps of the proposed algorithm, accelerate the convergent to the best value of the objective function without taking into account all points of the interpolation set. **Conclusion:** Any n-dimensional problem of finding a minimal value of a function, given by some values, can be converted to one dimensional problem easier in deal.

Key words: Quadratic interpolation, tabulated function, trust region, derivative free optimization

INTRODUCTION

Many optimization problems can be occur in practice, for example, in most of labs, one obtains data for certain phenomena and wants know that the point at which this phenomena is minimized. Also the problems which they arise in economic, social fields, engineering field and many others fields.

Many researchers had been construct several useful algorithms for dealing with this kind of problems. These algorithms, are classified as, algorithms use finite difference approximations of the objective functions derivatives^[2-4], the second class is pattern search methods which are based on the exploration of the variables' space using a well specified geometric pattern, typically a simplex^[8]. Finally, algorithms are based on the progressive building and updating of a model of the objective function^[9,10].

There is also a related class of "global modeling methods" that use design of interpolation models^[1,6,7].

In the following discussion we present analysis on which the proposed algorithm for finding the minimal point of tabulated function is based.

RESULTS AND DISCUSSION

Let I be denote to the set of points $x \in X \subset \mathbb{R}^n$ at which the function $f : X \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is given in the Table 1.

Table: 1 Initial interpolated set

x	x ₀	x ₁	x _j
f(x)	f(x ₀)	f(x ₁)	f(x _j)

Choose $x^k \in I$ such that $f(x^k) \leq f(x)$, for each $x \in I$ and consider real quadratic convex function $\phi(\alpha)$ such that

$$\phi(\alpha) = f(x^k + \alpha p),$$

Where:

$$p \in B \subset \mathbb{R}^n,$$

B is the trust region with radius Δ , i.e,

$$B = \left\{ x \in \mathbb{R}^n : \|x - x^k\| \leq \Delta \right\},$$

p is chosen such that $f(p) \geq f(x^k)$ and $f(p) < f(x)$ for each $x \in I, x \neq x^k$.

The quadratic convex function $\phi(\alpha)$ is constructed by interpolating the points $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$ such that the function f is given, at least, at two of α s. If f is known for each $\alpha_1, \alpha_2, \alpha_3$, then we can easy to construct $\phi(\alpha)$. If f is known for only two points of $\alpha_1, \alpha_2, \alpha_3$ then we can assume that the value of f at third point is z which

is calculated from the knowledge of differentiability of ϕ . If f is known only at one point of α , s, then we can consider one of the other two values (say $\phi(\alpha_2)$) is less than $\phi(\alpha_1)$ and $\phi(\alpha_3)$

Main results: The basic idea of this study is based on constructing a one dimensional real valued quadratic function $\phi(\alpha)$ by interpolating. The function $\phi(\alpha)$ is constructed such that $\phi(\alpha) = f(x^k + \alpha p)$ where x^k is the point in the set I at which the least value of f and the point p is chosen as indicated in above.

The following results show that the direction p is the descent direction and how we can build our algorithm.

Proposition 1: If $\bar{\alpha}$ is a minimal solution of the problem. $\min_{\alpha \in R} \phi(\alpha)$ and $p \in I$ is chosen as indicated before. Then

$$f(x^{k+1}) \leq f(x^k)$$

Where $x^{k+1} = x^k + \bar{\alpha} p$, i.e., p is the descent direction .

Proof: Since $\bar{\alpha}$ is a minimal solution of the problem $\min_{\alpha \in R} \phi(\alpha)$

So

$$\phi(\bar{\alpha}) \leq \phi(\alpha), \quad \forall \alpha \in R.$$

Thus

$$f(x^k + \bar{\alpha} p) \leq f(x^k + \alpha p), \quad \forall \alpha \in R.$$

Hence

$$f(x^k + \bar{\alpha} p) \leq f(x^k)$$

But $f(x^k) \leq f(x)$ for each $x \in I$, therefore the direction p is the decreasing direction of f .

Proposition 2: If $x^{k+1} = x^k + \bar{\alpha} p = \hat{x}$, then there is no other point $\hat{x} = x^k + \hat{\alpha} p \in I$ such that:

$$f(x^k + \hat{\alpha} p) \leq f(x^k).$$

Proof: Since $x^{k+1} = x^k$, so $\bar{\alpha} = \arg \min_{\alpha \in R} \phi(\alpha)$. $\phi(\alpha) = f(x^k + \alpha p^k)$. Also on the direction p^k there is no \hat{x} such that

$f(\hat{x}) < f(x^k + \bar{\alpha} p^k)$ because if there is such as that point, then there is $\hat{\alpha}$ such that $\hat{x} = x^k + \hat{\alpha} p^k$ and

$f(x^k + \hat{\alpha} p^k) < f(x^k + \bar{\alpha} p^k)$ i.e., $\phi(\hat{\alpha}) < \phi(\bar{\alpha})$ which is a contradiction.

Now, let \hat{p} be another descent direction, \hat{p} in the set I such that

$$f(\hat{p}) > f(p^k) > f(x^k).$$

Assume $\hat{x} = x^k + \hat{\alpha} \hat{p}$ is such that $f(\hat{x}) = f(x^k + \hat{\alpha} \hat{p}) < f(x^k + \bar{\alpha} p^k)$.

By constructing a function $\Psi(\alpha)$ such that $\Psi(\alpha) = f(x^k + \alpha \hat{p})$ we get

$$\Psi(\alpha) = \Psi(0) + \hat{\alpha} \hat{p} \nabla f(x^k) + \frac{\alpha^2}{2} \hat{p} \nabla^2 f(x^k) \hat{p} < f(x^k + \bar{\alpha} p^k)$$

Since \hat{p} is the descent direction and $\Psi(\alpha)$ is convex,

$$f(x^k) = \Psi(0) < f(x^k + \bar{\alpha} p^k)$$

Which is a contradiction.

Theorem: The sequence generated by

$$x^{k+1} = x^k + \bar{\alpha} p^k$$

is convergent, where $\bar{\alpha} = \arg \min_{\alpha \in R} \phi(\alpha)$ and P^k is in the trust region B^k

Proof: Since the trust region B^k is defined as

$$B_k = \left\{ x \in R^n : \|x - x^k\| \leq \frac{1}{k} \Delta \right\}.$$

This set is closed and bounded. Furthermore

$$x^{k+1} = x^k + \bar{\alpha} p^k \in B_k$$

Also,

$$x^{k+2} = x^{k+1} + \hat{\alpha} p^{k+1} \in B_{k+1},$$

Where B_{k+1} is also closed and $B_{k+1} \cap B_k \neq \emptyset$.

By repeating this process, we find the sequence $\{x^k\}$ is contained $B_k \supset \bigcap_{k=1}^m B_k$. Since B^k is closed, it contains the limit point of $\{x^k\}$, thus $\{x^k\}$ is convergent.

The Algorithm: From the previous discussion we can seek the following algorithm:

Initial step:

- 0.1:** Set $k=1$
- 0.2:** Choose $x_1 \in I$ such that $f(x_1) \leq f(x)$ for each $x \in I$
- 0.3:** Let Δ be the radius of the trust region B^1 ,

$$B_1 = \{x \in R^n : \|x - x_1\| \leq \Delta\}$$

- 0.4:** Choose $p_1 \in B_1$ such that $p_1 \in I$ and $f(p_1) \geq f(x_1)$,

$$f(p_1) < f(x), \forall x \in I, x \neq x_1$$

- 0.5:** Set $\varphi(\alpha) = f(x_1 + \alpha p_1), \alpha \in R$

The first Step:

- 1.1:** Choose $\alpha_1, \alpha_2, \alpha_3 \in R$ such that

$$\varphi(\alpha_1) = f(x_1 + \alpha_1 p_1)$$

$$\varphi(\alpha_2) = f(x_1 + \alpha_2 p_1)$$

$$\varphi(\alpha_3) = f(x_1 + \alpha_3 p_1).$$

- 1.2:** If $x_1 + \alpha_1 p_1, x_1 + \alpha_2 p_1, x_1 + \alpha_3 p_1 \in I$, then interpolate the points $\alpha_1, \alpha_2, \alpha_3$ to obtain a quadratic function $\phi(\alpha)$.

$$\bar{\alpha} = \arg \min_{\alpha \in R} \varphi(\alpha)$$

- 1.3:** Determine $\bar{\alpha}$ and go to step 2.

- 1.4:** If one of the points $x_1 + \alpha_i p_1 \notin I, i=1,2,3$ then put it equals z ,

$$\text{i.e., } \phi(\alpha_j) = f(x_1 + \alpha_j p_1) = z.$$

- 1.5:** Determine z that makes $\phi'(0) = 0$ and hence substitute z in ϕ to obtain $\phi(\alpha)$ and go to step 1-2.

- 1.6:** If two of the points $x_1 + \alpha_i p_1 \notin I$ (say $x_1 + \alpha_1 p_1, x_1 + \alpha_m p_1 \notin I$), then choose α_1, α_m such that

$$\phi(\alpha_1) < \phi(\alpha_i), \phi(\alpha_1) < \phi(\alpha_m), \phi(\alpha_m) = \phi(\alpha_i) \text{ or}$$

$$\phi(\alpha_m) < \phi(\alpha_i) < \phi(\alpha_1), \phi(\alpha_m) = \phi(\alpha_1)$$

and go to step 1.2.

The second step :

- 2.1:** If no improvement in value of f , then opposite the direction of P_1 to obtain $\phi(\alpha) = f(x_1 - \alpha p_1)$ and go to step 1. Otherwise go to step 3.

- 2.2:** If $x_1 - \alpha p_1$ does not improve the value of f , then let

$$\varphi(\alpha) = f(x_1 + \alpha(p_1 - x_1))$$

Such that $(p_1 - x_1) \in B_1$ and go to step 1. Otherwise go to step 3.

- 2.3:** If $(p_1 - x_1) \notin B_1$, then extend the radius of B_1 to contain $p_1 - x_1$ and go to step 1.

The third step:

- 3.1:** Inter $x_1^1 = (x_1 + \bar{\alpha} p_1 \text{ or } x_1 - \bar{\alpha} p_1 \text{ or } x_1 + \bar{\alpha}(p_1 - x_1))$ in the set I and reduce the radius of the trust region to become

$$B_2 = \left\{ x \in R^n : \|x - x_1^1\| \leq \frac{1}{2} \Delta \right\}.$$

Hence go to step 0.

- 3.2:** If $x_1^2 = x_1^1$, then stop and the minimal point is in the ball $B(x_1^2)$ with radius $\frac{\Delta}{2}$. Otherwise go to step 0.

Example: Determine the minimal point of the function $f(x)$ given by the Table 2.

Table 2: Initial set values of f

x	(-2, 0)	(1, 1)	(0, 1)	(1, 0)	(0, 2)	(1, 2)
$f(x)$	4	2	1	1	4	5

Choose $x_1 = (0, 1)$ and $\Delta = 2$, then

$$B_1 = \{(x, y) : \|(x, y) - (0, 1)\| \leq 2\}$$

Choose $p_1 = (1, 0)$

Let:

$$\phi(\alpha) = f[(0, 1) + \alpha(1, 0)] = f(\alpha, 1)$$

Take $\alpha_1 = 0, \alpha_2 = 1, \alpha_3 = -1$,
 $\phi(\alpha_1) = 1, \phi(\alpha_2) = 2$. set $\phi(\alpha_3) = z$

By using Maple Package we can obtain

$$\phi(\alpha) = \alpha^2 + 1 \text{ and } \bar{\alpha} = 0 = \arg \min_{\alpha \in \mathbb{R}} \phi(\alpha)$$

Thus $f(\alpha, 1)_{\alpha=0} = f(0, 1) = 1$, i.e., no improvement in the value of f , then consider the opposite direction $-p = (-1, 0)$

Therefore,

$$\phi(\alpha) = f(-\alpha, 1)$$

Which by interpolating the points

$$\begin{matrix} \alpha & : & \alpha_1 & \alpha_2 & \alpha_3 \\ j(\alpha) & : & 1 & 2 & z \end{matrix}$$

and by using Maple Package we get

$$\phi(\alpha) = \alpha^2 + 1$$

Which it has $\bar{\alpha} = 0$ as the minimal point and no improvement in the value of f . Therefore we will consider the direction $p_1 - x_1$, such that $p_1 - x_1 \in B_1$. Since $p_1 - x_1 = (1, -1) \notin B_1$, so extend the radius of B_1 to be $\Delta = 3$. In this case we option

$$\phi(\alpha) = f(\alpha, 1 - \alpha)$$

Since $\alpha_1 = 0$ implies $(0, 1) \in I$, then choose $\alpha_2 = \frac{1}{2}$ and $\alpha_3 = 1$ with letting corresponding values of $\phi(\alpha)$ as 0, 1 respectively.

By interpolating $\phi(\alpha)$ we obtain

$$\phi(\alpha) = 4\alpha^2 - 4\alpha + 1$$

Which it has $\bar{\alpha} = \frac{1}{2}$ as the minimal point. Then enter the point $x^2 = \left(\frac{1}{2}, \frac{1}{2}\right)$ in the set I with a corresponding value $\phi\left(\frac{1}{2}\right) = 0$, i.e., the set I becomes

$$\begin{matrix} x & : & (-2, 0) & (1, 1) & (0, 1) & \left(\frac{1}{2}, \frac{1}{2}\right) & (1, 0) & (0, 2) & (1, 2) \\ f(x) & : & 4 & 2 & 1 & 0 & 1 & 4 & 5 \end{matrix}$$

Reduce the trust region to become

$$B_2 = \{(x, y) : \|(x, y) - \left(\frac{1}{2}, \frac{1}{2}\right)\| \leq \frac{1}{2}\Delta = \frac{3}{2}\}$$

And choose $p_2 \in B_2, p_2 \in I \cup \left(\frac{1}{2}, \frac{1}{2}\right), p_2 = (1, 0)$. By using Maple we get $\phi(\alpha) = f\left(\alpha + \frac{1}{2}, \frac{1}{2}\right), \phi(\alpha) = \alpha^2$, which has the minimal point $\alpha = 0$.

Now no improvement, then consider the opposite direction $-p = (-1, 0)$ for which

$$\phi(\alpha) = f\left(\frac{1}{2} - \alpha, \frac{1}{2}\right)$$

and for the points

$$\begin{matrix} \alpha & : & -1 & 0 & 1 \\ \phi(\alpha) & : & 1 & 0 & 1 \end{matrix}$$

the function $\phi(\alpha) = \alpha^2$ and also no improvement.

Hence consider the direction $p_2 - x_2 = \left(\frac{1}{2}, \frac{-1}{2}\right)$. By using Maple we get $\phi(\alpha) = \alpha^2$ and $z = 1$ and no improvement in the consider three direction, then stop and the minimal point is in the ball $B\left(\frac{1}{2}, \frac{1}{2}\right)$ with radius $\frac{3}{2}$.

CONCLUSION

The presented algorithm in this paper enables us to find the point at which the value of objective function, in optimization, problem is the best value when some values are known at some points. The idea of this algorithm is interpolating the points to one dimensional quadratic function from which we can obtain the desired point.

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