Interpolating Operators for Multiapproximation

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Abstract: Problem statement: There are no simple definitions of operators for best multiapproximation and best one sided multiapproximation which work for any measurable function in L_p for, $p>0$. This study investigated operators that are good for best multiapproximation and best one sided multiapproximation. **Approach:** We first introduced some direct results related to the approximation problem of continuous functions by Hermit-Fejer interpolation based on the zeros of Chebyshev polynomials of the first or second kind in terms of the usual modulus of continuity. They were then improved to spaces L_p for $p<1$, in terms of the first order averaged modulus of smoothness. However, because this suffers from convergence problems, we improved and generalized these direct estimations by defining an interpolating multivariate operator $H_n(f)$ of measurable functions, that operator based on the zeros of Chepyshev polynomials of the first kind and prove that for any measurable function defined on $L_p[-1,1]^d$ the sequence $H_n(f)$ converges uniformly to f. **Results:** The resulting operators were defined for functions f such that $f^{(k)}, k = 0,1,\dots$ is of bounded variation. Then, the order of best onesided trigonometric approximation to bounded measurable functions in terms of the average modulus of smoothness was characterized. Estimates characterizing the order of best onesided approximation in terms of the k-th averaged modulus of smoothness for any function in spaces L_p, $p<\infty$ were obtained. In our research we also approximated one sidedly these measurable functions in $L_p[-1,1]^d$ by defining a new operator for onesided approximation and prove a direct theorem for best one sided multiapproximation in terms of the first order averaged moduli of smoothness. **Conclusion:** The proposed method successfully construct operators for best multi approximation and best one sided multiapproximation for any measurable function in L_p for, $p>0$.

Key words: Interpolating operator, best approximation, onesided approximation, multiapproximation

INTRODUCTION

In Jishan and Ziyu (1989), introduced some results related with the approximation problem of continuous functions by Hermit-Fejer interpolation based on the zeros of Chebyshev polynomials of the first or second kind. And they proved:

If $f \in C[-1,1]$, then:

$$\left| H_n^1(f,x) - f(x) \right| \le c \left(\frac{1}{n} \left| T_n(x) T_{n-1}(x) \right| \left| f'(x) \right| \right.$$

$$+ \frac{1}{n} \left| T_n(x) \right| \omega \left(f', \frac{\left| T_n(x) \right|}{n} \right)$$

$$+ \frac{1}{n} \left| T_n(x) \right|^2 \sum_{k=1}^{n} \left(\frac{\sqrt{1+x^2}}{k} + \frac{1}{k^2} \right) \omega \left(f', \frac{\sqrt{1+x^2}}{k} + \frac{1}{k^2} \right)$$

where, $\omega \left(f', \frac{\sqrt{1+x^2}}{k} + \frac{1}{k^2} \right)$ is the usual modulus of smoothness for f'.

Jassim and Tahir (1996) introduced a research to improve the above estimation to the spaces $L_p[-1,1]$ for $p \ge 1$, they proved:

$$\left\| H_n^1(f,x) - f(x) \right\|_p \le \frac{c}{n} \sum_{k=1}^{n} \tau \left(f', \frac{k\sqrt{1+x^2}}{n} + \left(\frac{k}{n} \right)^2 \right)_p$$

where, $\tau \left(f, \frac{k\sqrt{1+x^2}}{n} + \left(\frac{k}{n} \right)^2 \right)_p$ is the averaged modulus of smoothness for f of the first order.

But these convergence are all not uniformly and the right hand side may not converge to zero. For this reason we shall introduce an improvement and a generalization for the estimations above. That by defining an interpolating multivariate operator $H_n(f)$ of the measurable functions, that operator based on the zeros of Chepyshev polynomials of the first kind and prove that for any measurable function defined on $L_p[-1,1]^d$ the sequence $H_n(f)$ converges uniformly to f.

MATERIALS AND METHODS

Theorem 1: For any measurable function f in $L_p[-1,1]^d$ $0<p<\infty$ we have:

$$\|f - H(f)\|_p \le c(p,d)\tau\left(f,\frac{1}{n}\right)_p$$

where, $c(p, d)$ is a constant depending on p and d only.

The first authors on onesided approximation, Freud (1955) and Ganelius (1956), used operators for constructing polynomials of one sided approximation. These operators were defined for functions f such that $f^{(k)}, k = 0,1,\ldots$ is of bounded variation. Also Andreev *et al.* (1979); Popov (1977) and Popov and Andrev (1978) characterized the order of best onesided trigonometric approximation to bounded measurable functions in terms of the average moduli. After that Stojanova (1988); Popov and Andrev (1978) and Hristov and Ivanov (1988; 1990a; 1990b) obtained estimates characterizing the order of best onesided approximation in terms of the kth averaged modulus of smoothness for any function in the spaces L_p, $p<\infty$. In this study we also approximate one sidedly these measurable functions in $Lp[-1,1]^d$ by defining a new operator for onesided approximation and prove the following result:

Theorem 2: For any bounded and measurable functions in $L_p[-1,1]^d$ $0<p<\infty$:

$$\|H^+(f) - H^-(f)\|_p \le c(p,d)\tau\left(f,\frac{1}{n}\right)_p$$

Let $T_n(x) = \cos(n\cos^{-1}(x))$, $x \in [-1,1]$ be the first kind Chebyshev polynomial and let: $x_k = \cos\theta_k, \theta_k = \frac{(2k-1)\pi}{2n}, k = 1, 2, \ldots,n$ be its zeros. Now for a function $f \in C[-1,1]^d$, we denote by $H_{2n-1}^d(f,x)$ the interpolating polynomial based on $\{x_k\}_{k \in N_n^d}$ defined by:

$$H_{2n-1}^d(f,x) = H(f,x) = \sum_{k \in N_n^d} f(x_k)h_k(x)$$

where, $x \in [-1,1]$, $(x_k,\ldots,x_{kd}) = (\cos\theta_{k_1},\theta_{k_d})$, $\theta_{k_s}\frac{(2k_s-1)\pi}{2n}$ $x(x_1,\ldots,x_d) = (\cos\theta_1,\ldots,\cos\theta_d) = \Phi(\theta)$, $\theta(\theta_1,\ldots,\theta_d)$, $\theta_s \in [0,\pi]$, $k = (k_1,\ldots k_d)$, $\Omega = [0,\pi]^d$, $h_k(x) = I_{k_1}(x_1),\ldots,I_{k_d}(x_d)$

For $x \in R^d$, let us define $|x| = \max\{|x_i|, i = 1,\ldots,d\}$,

$$I_{k_s}(x_s) = (1 - x_s x_{k_s})^{\ell+1}\left(\frac{T_n(x_s)}{n(x_s - x_{k_s})}\right)^{2\ell+2}, \ S = 1,\ldots,d \text{ and } \ell$$

$\in Z$. It is clear that $H(f,.)$ is a polynomial of degree $(2n-1)(\ell+1)d$. $T_n(x_s) = \cos(n\cos^{-1}(x_s))$. We can easily prove that for each k_s, $I_{k_s}(x_{k_s})$ and therefore:

$$H(f, x_k) = (f(x_k)) \tag{1}$$

and

$$D^{\alpha}H(f, x_k) = 0 \tag{2}$$

Where:
$D^{\alpha} = D_1,D_2,\ldots D_n$
$\alpha = (\alpha_1,\ldots, \alpha_d), \alpha_1 = \ldots = \alpha_d = 1$
$D_1 = \frac{\partial^{\alpha_1}H}{\partial x_{\alpha_1}}$
$D_2 = \frac{\partial^{\alpha_2}H}{\partial x_{\alpha_2}},\ldots,D_d = \frac{\partial^{\alpha_d}H}{\partial x_{\alpha_d}}$

In the light of (1) and (2) we have this polynomial is uniquely characterized, so that $H(1,k)\sum_{k \in N_n^d} h_k(x) = 1$.

Let $\Delta_h^k f(x)$ denote the kth finite difference with step h of f in the point x. We denote by:

$$\omega_k(f,x,\delta) = \sup\left\{\left|\Delta_h^k f(y)\right| : y, y + kh \in N\left(\frac{k\delta}{2},x\right)\right\}$$

where, $N\left(\frac{k\delta}{2},x\right) = \left\{y \in R^d : \|y - x\| < \frac{k\delta}{2}\right\}, d,k \in N, \delta > 0$, the local modulus of f. Two global moduli of the function f will be used, the usual modulus of smoothness:

$$\omega_k(f,\delta)_{L_{p([-1,1]^d)}} = \sup\left\{\left\|\Delta_h^k f(.)\right\|_{L_{p([-1,1]^d)}} : |h| \le \delta\right\}$$

and the average modulus of smoothness:

$$\tau_k(f,\delta)_{L_{p([-1,1]^d)}} = \left\|\omega_k(f,\ldots,\delta)\right\|_{L_{p([-1,1]^d)}}$$

The properties of ω_k are assumed to be known. Some properties of τ_k are given in (Sendov and Popov, 1991).

In the sequel we denote by c an absolute constant which may vary from one occurrence to another even in the same line. Similarly c(.) will denote a constant which depends on a specific parameter but may change from one occurrence to another.

RESULTS AND DISCUSSION

The first aim of this part is to prove the following direct theorem for best multi approximation:

Theorem 3: For any measurable function f in $L_p[-1,1]^d, \infty < p \le 1$, there exists a polynomial H(f) such that:

$$\|f - H(f)\|_p \le c(p,d)\tau\left(f,\frac{1}{n}\right)_{L_{p([-1,1]^d)}}$$

To prove our third theorem we need the following assertions:

Lemma 1: Let $0 < \lambda t \le \text{diam}\Omega$, then Hristov and Ivanov (1988):

$$\tau_k(f,\lambda t)L_{p(\Omega)} \le c\lambda^{\ell}\tau_k(f,t)L_{p(\Omega)}$$

where, $\ell = \max\{k,m\}$ and $m = \left[\dfrac{d}{p}\right] + 1$ ([.]-integer part).

Lemma 2: See Hristov and Ivanov (1988):

$$\int_{-\pi}^{\pi}\left(\frac{\sin\frac{nv}{2}}{\sin\frac{v}{2}}\right)^{2\ell+2}|v|^r\,dv \le cn^{-r}, \quad r = 0,1,....2\ell$$

Lemma 3:

$$\int_{0}^{\pi}\frac{1}{n^{2\ell+1}}\left(\frac{\sin\frac{nv}{2}}{\sin\frac{v}{2}}\right)dv \le \frac{c(\zeta)}{n^{2\ell+1}}, \quad \xi \in [0,1]$$

Proof: The lemma is easily proved by the properties of the trigonometric functions and the Bonnets (Gupta and Rani 1989) mean value theorem.

Lemma 4: Sendov and Popov (1991). For any bounded and measurable function f on [a,b] we have:

$$\int_{a}^{b}f(x)dx \approx \frac{b-a}{n}\sum_{i=1}^{n}f(x_i), \quad x_i = a + (b-a)(2i-1)2n$$

The proof of Theorem 3:

$$|f(x) - H(f,x)| = \left|\sum_{k \in N_n^d}f(x) - f(x_k))H_k(x)\right|$$

Let $cox\theta_s$ and $(\cos\theta_1,...,\cos\theta_d) = \Phi(\theta)$, $x \in [-1,1]^d$ and $\theta \in [\theta,\pi]^d$. Then:

$$|f(x) - H(f,x)|$$

$$\le \sum_{k \in N_n^d}|f(x) - f(x_k)|\prod_{s=1}^{d}\left|1 - x_s x_{k_s}\right|^{\ell+1}\left(\frac{\cos n\theta_s}{n(\cos\theta_s - \cos\theta_{k_s})}\right)^{2\ell+2}$$

Then since $\sin\theta_s \ge 0$, $\sin\theta_{k_s} \ge 0$, $\cos n\theta_{k_s} = 0$, so:

$$|f(x) - H(f,x)|$$

$$\le c\sum_{k \in N_n^d}|f(x) - f(x_k)|\prod_{s=1}^{d}\left|1 - \cos(\theta_s - \theta_{k_s})\right|^{\ell+1}$$

$$\left(\frac{\sin\left(\frac{n(\theta_s - \theta_{k_s})}{2}\right)}{n\sin\left(\frac{\theta_s - \theta_{k_s}}{2}\right)}\right)^{2\ell+2}$$

$$\le c\sum_{k \in N_n^d}|f(x) - f(x_k)|\prod_{s=1}^{d}\left(\frac{\sin\left(\frac{n(\theta_s - \theta_{k_s})}{2}\right)}{n\sin\left(\frac{\theta_s - \theta_{k_s}}{2}\right)}\right)^{2\ell+2}$$

Let us assume that:

$$J_{\ell,n}(\theta_s - \theta_{k_s}) = \frac{c}{n^{2\ell+1}}\left(\frac{\sin\left(\frac{n(\theta_s - \theta_{k_s})}{2}\right)}{n\sin\left(\frac{\theta_s - \theta_{k_s}}{2}\right)}\right)$$

and

$$I_{\ell,n}(\theta - \theta_k) = J_{\ell,n}(\theta_1 - \theta_{k1})\,....\,J_{\ell,n}(\theta_d - \theta_{kd})$$

Then:

$$|f(x) - H(f,x)| \le c\left(\frac{1}{n}\right)^{d}|f(\Phi(\theta)) - f(\Phi(t))|J_{\ell,n}(\theta_k)$$

Then by Lemma 4 we have:

$$|f(x) - H(f,x)| \le c \int_\Omega |f(\Phi(\theta)) - f(\Phi(t))| J_{\ell,n}(t) dt$$

Thus:

$$\|f(x) - H(f,x)\|_p \le c \left(\int_\Omega \left(\int_\Omega |f(\Phi(\theta)) - f(\Phi(t)| J_{\ell,n}(t) dt \right)^p d\theta \right)^{\frac{1}{p}}$$

$$\le c \left(\int_\Omega \left(\int_\Omega \omega(f(\Phi(\theta)), \theta, |\theta - t|) \right)^p d\theta \right)^{\frac{1}{p}} J_{\ell,n}(t) dt$$

$$= c \int_\Omega \tau(f(\Phi(\theta)), |\theta - t|) L_{p(\Omega)} J_{\ell,n}(t) dt$$

Lemma 1 and Lemma 2, lead to:

$$\|f(x) - H(f,x)\|_p \le c(\zeta)\tau\left(f(\Phi(\theta)), \frac{1}{n} \right)_{L_{p(\Omega)}}$$

$$= c(\zeta)\tau\left(f(x), \frac{1}{n} \right)_{L_p\left([-1,1]^d\right)}$$

For the case $0 < p < 1$, we can prove theorem1 by follows the same lines above and using the inequality (Carathers, 2005):

$$\|f + g\|_p^p \le \|f\|_p^p + \|g\|_p^p$$

In the second result of present study we construct a one-sided operator using $H(f, x)$ as follows:

$$H^{+,-}(f,x) = H(f,x) \pm \sum_{k \in N_n^d} \psi(x) \sup_{\left|r(y) - \frac{\pi(2j-1)}{2n}\right| \le \frac{\pi}{2n}} |f(y) - H(f,y)|$$

Where:
$r(y) = (\cos^{-1}y_1, \ldots, \cos^{-1}y_d)$
$y_1 = \cos\theta_1, \ldots, y_d = \cos\theta_d$

$$\psi(x_s) = \prod_{i=1}^d \left(\frac{T_n(x_s)}{n(x_s - x_{si})} \right)^2 (1 - \omega_i z_i)^\ell, T_n(x_s) = \cos(n\cos^{-1}x_s),$$

$$\omega_i = \cos\frac{\pi}{n}, z_i = \cos\frac{\pi}{2n}, x_s \in [-1,1] = \eta,$$

$$x_{si} = \cos\theta_{si}, \theta_{si} = \frac{(2s_i - 1)\pi}{2n}$$

and

$$|x| = \sup_{1 \le i \le d} |x_i|$$

This operator is a one-sided operator, since:

- The polynomials H^{+-} (f, x) of degree $(2n - 1)(l + 1)d$
- H^- (f, x)$\le$f(x)$\le$H$^+$(f, x),x$\in$[-1,1]^d = η. Because of the positivity of ψ (x$_s$), for each s = 1,2,…,d

Then we introduce our second result.

Theorem 4: For any bounded function f in L_p $[-1,1]$ = $\eta, \infty < p \le 1$, there exist two polynomials $H^+(f)$ and $H^-(f)$ such that:

$$\|H^+(f) - H^-(f)\|_p \le c(p,d)\tau\left(f, \frac{1}{n} \right)_{L_{p(\eta)}}$$

For the proof of this theorem we need the following Lemma:

Lemma 5:

$$\psi(x_s) = \prod_{i=1}^d \left(\frac{T_n(x_s)}{n(x_s - x_{si})} \right)^2 (I - \omega_i z_i)^\ell \le$$

$$c \prod_{i=1}^d \left(\frac{\sin\left(\frac{n(\theta_s - \theta_{s_i})}{2} \right)}{n\sin\left(\frac{(\theta_s - \theta_{s_i})}{2} \right)} \right)^2 \sin\left(\frac{\pi}{2n} \right)^{2\ell} \le \left(\frac{\pi}{2n} \right)^{2\ell}$$

Proof: Let $x_{si} = \cos\theta_{s_i}, x_s = \cos\theta_s \theta_{s_i} = \frac{(2_{s_i} - 1)\pi}{2n}$, then

$$\psi(x_s) = \prod_{i=1}^d \left(\frac{\cos\theta_s}{n(\cos\theta_s - \cos\theta_{s_i})} \right)^2 \left(1 - \cos\frac{\pi}{2}\cos\frac{\pi}{2n} \right)^\ell \cos\theta_{si} =$$

0 implies:

$$\psi(x_s) = \prod_{i=1}^d \left(\frac{\cos\theta_s - \cos n\theta_{si}}{n(\cos\theta_s - \cos\theta_{si})} \right)^2 \left(1 - \cos\frac{\pi}{2}\cos\frac{\pi}{2n} \right)^\ell$$

$$\le \prod_{i=1}^d \left(\frac{\sin\left(\frac{n(\theta_s - \theta_{si})}{2} \right)}{n\sin\left(\frac{(\theta_s - \theta_{si})}{2} \right)} \right)^2 \left(1 - \cos\frac{\pi}{2}\cos\frac{\pi}{2n} + \sin\frac{\pi}{n}\sin\frac{\pi}{2n} \right)^\ell$$

$$< \prod_{i=1}^d \left(\frac{\sin\left(\frac{(\theta_s - \theta_{si})}{2} \right)}{n\sin\left(\frac{(\theta_s - \theta_{si})}{2} \right)} \right)^2 \left(1 - \cos\left(\frac{\pi}{2} + \frac{\pi}{2n} \right) \right)^\ell$$

$$= \prod_{i=1}^d \left(\frac{\sin\left(\frac{(\theta_s - \theta_{si})}{2} \right)}{n\sin\left(\frac{(\theta_s - \theta_{si})}{2} \right)} \right)^2 \sin^{2\ell}\frac{\pi}{2n} \le 2^d \sin^{2\ell d}\frac{\pi}{2n}$$

which completes the proof.

Proof of Theorem 4:

$$\sup_{\left|r(y)-\frac{\pi(2j-1)}{2n}\right|\le\frac{\pi}{2n}}\left|f(y)-H(f,y)\right|\le\left(\frac{c}{n}\right)^d\sup_{\left|r(y)-\frac{\pi(2j-1)}{2n}\right|\le\frac{\pi}{2n}}$$

$$\sum_{k\in N_n^d}\left|f(y)-f(x_k)\right|J_{\ell,n}\left(\frac{(2k-1)\pi}{2n}\right)$$

Now let $x_{ks,}=\cos\theta_k, \theta_k=\frac{(2k_s-1)\pi}{2n}$, $r(y)=$ $(\cos^{-1}y_1,\ldots,\cos^{-1}y_d)$, $y_1=\cos\theta'_1,\ldots,y_d=\cos\theta'_d$. Then:

$$\sup_{\left|r(y)\frac{\pi(2j-1)}{2n}\right|\le\frac{\pi}{2n}}\left|f(y)-H(f,y)\right|\le\left(\frac{c}{n}\right)^d\sup_{\left|r(y)\frac{\pi(2j-1)}{2n}\right|\le\frac{\pi}{2n}}$$

$$\int_{\Omega=[0,\pi]^d}\left|f(\Phi(\theta'))-f(\Phi(t))\right|J_{\ell,n}(t)dt$$

$$\le\left(\frac{c}{n}\right)^d\sup_{\left|r(y)\frac{\pi(2j-1)}{2n}\right|\le\frac{\pi}{2n}}\int_{\Omega}\omega\left(f\circ\Phi,\frac{\pi(2j-1)}{2n},\frac{\pi}{2n}\right)J_{\ell,n}(t)dt$$

$$\le\left(\frac{c}{n}\right)^d\sup_{\left|r(y)\frac{\pi(2j-1)}{2n}\right|\le\frac{\pi}{2n}}\int_{\Omega}\omega\left(f\circ\Phi,\frac{\pi(2j-1)}{2n},\frac{\pi}{2n}+|t|\right)J_{\ell,n}(t)dt$$

Then:

$$\left\|H^+(f)-H^-(f)\right\|_p=2\left\|\sum_{j\in N_n^d}\psi(x)\sup_{\left|r(y)\frac{\pi(2j-1)}{2n}\right|\le\frac{\pi}{2n}}\left|f(y)-H(f,y)\right|\right\|_p$$

$$\le2\left\|\sum_{j\in N_n^d}\psi(x)\int_{\Omega}\psi(x)\omega\left(f\circ\Phi,\frac{\pi(2j-1)}{2n},\frac{\pi}{2n}+|t|\right)J_{\ell,n}(t)dt\right\|_p$$

Then realize on Lemma 5, to obtain:

$$\left\|H^+(f)-H^-(f)\right\|_p=2\left\|\int_{\Omega}\sum_{j\in N_n^d}\left(f\circ\Phi,\frac{\left(\frac{\pi}{2n}\right)\omega}{\frac{\pi(2j-1)}{2n},\frac{\pi}{2n}+|t|}\right)J_{\ell,n}(t)dt\right\|_p$$

$$\le c\left\|\int_{\Omega}\int_{\Omega}\left(\frac{\pi}{2n}\right)^{d\ell}\omega\left(f\circ\Phi,\theta,\frac{\pi}{2n}+|t|\right)d\theta J_{\ell,n}(t)d\right\|_p$$

$$\le c(p,d)\left(\frac{1}{n}\right)^{d\ell}\left\|\int_{\Omega}\left\|\omega\left(f\circ\Phi,\theta,\frac{\pi}{2n}+|t|\right)\right\|_{L_{p(\Omega)}}J_{\ell,n}(t)dt\right\|_p$$

$$\le c(p,d)\left(\frac{1}{n}\right)^{d\ell}\left\|\int_{\Omega}\tau\left(f\circ\Phi,\theta,\frac{\pi}{2n}+|t|\right)_{L_{p(\Omega)}}J_{\ell,n}(t)dt\right\|_p$$

By virtue of Lemma 1 and Lemma 2, we immediately have:

$$\left\|H^+(f)-H^-(f)\right\|_p\le c(p,d)\left(\begin{array}{l}\tau\left(f\circ\Phi,\frac{\pi}{2n}\right)_{L_{p(\Omega)}}J_{\ell,n}(t)dt+\\\int_{\Omega}\int_{\Omega}\tau\left(f\circ\Phi,\frac{|t|n}{n}\right)_{L_{p(\Omega)}}J_{\ell,n}(t)dt\end{array}\right)$$

$$\le c(p,d)\tau\left(f\circ\Phi,\frac{\pi}{n}\right)_{L_{p(\Omega)}}+\int_{\Omega}n^{\ell}|t|^{\ell}\tau\left(f\circ\Phi,\frac{|t|n}{n}\right)_{L_{p(\Omega)}}$$

$$J_{\ell,n}(t)dt\le c(p,d)\tau\left(f\circ\Phi,\frac{1}{n}\right)_{L_{p(\Omega)}}\le(p,d)\left(f,\frac{1}{n}\right)_{L_{p(\eta)}}$$

CONCLUSION

In this study, we successfully constructed operators for best multiapproximation and best one sided multiapproximation for the study of any measurable function in L_p for, p>0.

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