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On the Derived Subgroups of Some Finite Groups

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Abstract: Problem statement: In this study we focus on the derived subgroup of nonabelian 3-generator groups of order p^3q, where p and q are distinct primes and p < q. Our main objective is to compute the derived subgroup for these groups up to isomorphism. **Approach:** In a group G, the derived subgroup G' = [G, G] is generated by the set of commutators of G, K (G) = {[x, y]| x, y ∈ G} and introduced by Dedekind. The relations of the group are used to compute the derived subgroup. **Results:** The results show that the derived subgroup of nonabelian 3-generator groups of order p^3q is a cyclic group, Q_8 or A_4. **Conclusion/Recommendations:** The problem can be considered to compute the derived subgroup of these groups without the use of the relations.

Key words: Derived subgroup, sylow theorems, finitely generated group

INTRODUCTION

Miller (1898) introduced the derived subgroup G' of a group G as the subgroup generated by K (G) = {[x, y]| x, y ∈ G}, the set of commutators of G. According to Miller, commutators [x, y] were introduced by Dedekind a few years earlier. Commutators can act as a tool in all of group theory. For example, commutators can be used to compute Schur multiplier, Schur multiplier of a pair and nonabelian tensor squares of groups.

Basic definitions and theorems: Includes some definitions and results on the derived subgroups of nonabelian groups.

Definition 1: Hungerford (1997) let G be a group and X a subset of G. Let $\{H_i \mid i \in I\}$ be the family of all subgroups of G which contains X. Then $\cap H_i$ is called the subgroup of G generated by the set **X** and is denoted by < X >.

Theorem 2: Hungerford (1997) let G be a group and X a non empty subset of G. Then the subgroup < X > generated by X consists of all finite product finite product $a_1^{n_1}a_2^{n_2}a_3^{n_3}\ldots a_t^{n_t}$ ($a_i \in X$, $n_i \in Z$). In particular for every a ∈ G, $\langle a \rangle = \{a^n \mid n \in Z\}$.

Definition 3: Hungerford (1997) let G is a group. The subgroup of G generated by the set $\{x^{-1}y^{-1}xy \mid x, y \in G\}$ is called the derived subgroup of G and denoted by G'.

Let G be a group and let $G^{(1)}$ be G'. Then for i ≥ 1, define $G^{(i)} = G^{(i-1)'}$. The notation $G^{(i)}$ is called the ith derived subgroup of G. This gives a sequence of subgroups of G, each normal in preceding one: $G > G^{(1)} > G^{(2)} > \cdots$. Actually each $G^{(i)}$ is a normal subgroup of G.

Burnside (1911) classified all finite groups of order p^2q and Western (1898) obtained the classification of groups of order p^3q, where p and q are distinct primes.

The classification of all nonabelian 2-generator groups of order p^3q is given in the following theorem.

Theorem 4: Western (1898) Let G be a nonabelian 2-generator group of order p^3 q, where p and q are distinct primes and p < q. Then G is exactly one group of the following types Eq. 1-6:

$$G = \langle A, Q \mid A^8 = Q^q = 1, A^{-1}QA = Q^{-1} \rangle; q \equiv 1 \pmod 2 \tag{1}$$

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$$G = \langle A, Q \mid A^8 = Q^q = 1, A^{-1}QA = Q^a \rangle \quad (2)$$

where, a is any primitive root of a4 ≡ 1(mod q), q ≡ 1 (mod 4):

$$G = \langle A, Q \mid A^8 = Q^q = 1, A^{-1}QA = Q^a \rangle \quad (3)$$

where, a is any primitive root of a8 ≡ 1 (mod q), q ≡ 1 (mod 8):

$$G = \langle A, Q \mid A^{p3} = Q^q = 1, A^{-1}QA = Q^a \rangle \quad (4)$$

where, a is any primitive root of ap ≡ 1 (mod q), q ≡ 1 (mod p):

$$G = \langle A, Q \mid A^{p3} = Q^q = 1, A^{-1}QA = Q^a \rangle \quad (5)$$

where, a is any primitive root of ap2 ≡ 1 (mod q), q ≡ 1 (mod p^2):

$$G = \langle A, Q \mid A^{p2} = Q^q = 1, A^{-1}QA = Q^a \rangle \quad (6)$$

where, a is any primitive root of $a^{p3} \equiv 1 \pmod{q}$, $q \equiv 1 \pmod{p^3}$.

Theorem 5: Rashid *et al*. (2010) Let G be a nonabelian 2-generator group of order p^3q, where p and q are distinct primes and p < q. Then, $G' \cong C_q$, finite cyclic group of order q.

In this study, we focus on the derived subgroups of nonabelian 3-generator groups of order p^3q where p and q are distinct primes and p < q.

The classification of all nonabelian 3-generator groups of order p^3q is given in the following theorem.

Theorem 6: Western (1898) Let G be a nonabelian 3-generator group of order p^3q, where p and q are distinct primes and p < q. Then G is exactly one group of the following types Eq. 7-21:

$$G = \langle A, B, Q \mid A^4 = B^2 = Q^q = 1, BAB = A^{-1}, AQ = QA, BQ = QB \rangle \quad (7)$$

$$G = \langle A, B, Q \mid A^4 = B^4 = Q^q = 1, B^2 = A^2, B^{-1}AB = A^{-1}, AQ = QA, BQ = QB \rangle \quad (8)$$

$$G = \langle A, B, Q \mid A^4 = B^2 = Q^q = 1, AB = BA, AQ = QA, BQB = Q^{-1} \rangle \quad (9)$$

$$G = \langle A, B, Q \mid A^4 = B^2 = Q^q = 1, AB = BA, A^{-1}QA = Q^{-1}, BQ = QB \rangle \quad (10)$$

$$G = \langle A, B, Q \mid A^4 = B^2 = Q^q = 1, BAB = A^{-1}, AQ = QA, BQB = Q^{-1} \rangle \quad (11)$$

$$G = \langle A, B, Q \mid A^4 = B^2 = Q^q = 1, BAB = A^{-1}, A^{-1}AQ = Q^{-1}, BQ = QB \rangle, q \equiv 1 (\mathrm{mod} 2) \quad (12)$$

$$G = \langle A, B, Q \mid A^4 = B^4 = Q^q = 1, B^2 = A^2, B^{-1}AB = A^{-1}, AQ = QA, B^{-1}QB = Q^{-1} \rangle \quad (13)$$

$$G = \langle A, B, Q \mid A^4 = B^2 = Q^q = 1, AB = BA, A^{-1}QA = Q^a, BQ = QB \rangle$$

where, a is any primitive root of:

$$a^4 \equiv 1(\mathrm{mod} q) \mathrm{and} q \equiv 1(\mathrm{mod} 4) \quad (14)$$

$$G = \langle A, B, Q \mid A^4 = B^4 = Q^3 = 1, B^2 = A^2, B^{-1}AB = A^{-1}, Q^{-1}AQ = B, Q^{-1}BQ = AB \rangle \quad (15)$$

$$G = \langle A, B, Q \mid A^4 = B^4 = Q^3 = 1, BAB = A^{-1}, Q^{-1}A^2B = B, A^{-1}QA = Q^2A^2B \rangle \quad (16)$$

$$G = \langle A, B, Q \mid A^{p2} = B^p = Q^q = 1, B^{-1}AB = A^{p+1}, AQ = QA, BQ = QB \rangle \quad (17)$$

$$G = \langle A, B, Q \mid A^{p2} = B^p = Q^q = 1, AB = BA, AQ = QA, B^{-1}QB = Q^a \rangle$$

where, a is any primitive root of:

$$a^p \equiv 1(\mathrm{mod} q) \mathrm{and} q \equiv 1(\mathrm{mod} p) \quad (18)$$

$$G = \langle A, B, Q \mid A^{p2} = B^p = Q^q = 1, AB = BA, A^{-1}QA = Q^a, BQ = QB \rangle$$

where, a is any primitive root of:

$$a^p \equiv 1(\mathrm{mod} q) \mathrm{and} q \equiv 1(\mathrm{mod} p) \quad (19)$$

$$G = \langle A, B, Q \mid A^{p2} = B^p = Q^q = 1, B^{-1}AB = A^{p+1}, AQ = QA, B^{-1}QB = Q^b \rangle$$

where, a is any primitive root of $a^p \equiv 1$

$$(\text{mod}\,q), q \equiv 1(\text{mod}\,p)\ \text{and}\ b = a,\ a^2,\ \ldots,\ a^{p-1} \quad (20)$$

$$G =< A,\ B,\ Q \mid A^{p2} =\ B^{p} =\ Q^{q} = 1, \\ AB\ =\ BA, A^{-1}QA\ =\ Q^{a},\ BQ\ =\ QB >$$

where, a is any primitive root of $a^{p2} \equiv 1$:

$$(\text{mod}\,q)\ \text{and}\,q \equiv 1(\text{mod}\,p^2) \quad (21)$$

Main Result:

Theorem 7: Let G be a nonabelian 3-generator group of order p^3 q, where p and q are distinct primes and p < q. Then $G' \cong C_2$, Cq, C_{2q}, C_p, C_{pq}, Q_8 or A_4, where Q_8, A_4 are quaternion and alternating groups, respectively.

Proof: By Theorem 6, G has 15 types. If G is a group of type 6.1, then G has three generators A, B and Q and relations BAB = A^{-1}, AQ = QA and BQ = QB. For this group we can obtain the following relations:

- $A^iQ^j = Q^jA^i$; for all i, j ∈ Z
- $B^iQ^j = Q^jB^i$; for all i, j ∈ Z
- AB = BA^{-1} , $A^2B = BA^2$, A^3B = BA
- [A, B] = A^2, [A^2 , B] = 1

Then by mentioned relations for all x, y ∈ G, [x, y] = 1 or A^2. Therefore, $G' = \{1, A^2\}$, that is, $G' \cong C_2$.

The proof of the second type is similar to the first type.

To compute the derived subgroup for a group of type 6.3, by relations AB = BA, AQ = QA, BQB = Q^{-1} and [Q^k, B] = Q^{-2k}, we can obtain that $G' \cong C_q$.

The proof of types 6.4, 6.8, 6.12, 6.13 and 6.15 is similar to that type of 6.3.

For type 6.5, G ≅ D_{4q}, then $G' \cong C_{2q}$.

Let G be a group of type 6.6, then by relation A^{-1} AQ = Q^{-1} it is clear that | G′ | ≥ pq and relation, BAB = A^{-1} shows that 1, $A^2 \in G'$. Thus | G′ | = 2q and G′ ≅ < BQ >, that is, $G' \cong C_{2q}$.

For proving 6.7, we can use the method that we used in type 6.6.

For a group of type 6.9, G ≅ SL (2, 3), where SL (2, 3) =< a, b, c | a^3 = b3 = c^2 = abc>. So $G' \cong Q_8$.

To compute G′ for a group of type 6.10, by the number of generators and relations it is an immediate consequence that G ≅ S4. Therefore, $G' \cong A_4$.

Let G be a group of type 6.11, then relations A^{p2} = B^p = Q^q = 1, $B^{-1}AB = A^{p+1}$, AQ = QA, BQ = QB show that G′ is isomorphic to C_p.

Finally, for a group of type 6.14, the relations A^{p2} = B^p = Q^q = 1, $B^{-1}AB = A^{p+1}$, AQ = QA, B^{-1}Q B= Q^b show that | G′ | = pq and by computing the commutators, G′ is a cyclic group of order pq. □

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