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On Applications of Differential Subordination and Differential Operator

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Abstract: Problem statement: The object of this study is to obtain certain differential subordinations. **Approach:** Here we use known generalized differential operator given by Darus and Ibrahim and well known lemmas given by Miller and Mocanu. **Results:** We will pose several results on subordination theorems. **Conclusion:** Many other results can be obtained by using the operator defined.

Key words: Differential operator, analytic functions, subordination, superordination

INTRODUCTION

Let H be the class of analytic functions in $U := \{z \in C : |z| < 1\}$ and $H[a,n]$ be the subclass of H consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$. Let A be the subclass of H consisting of functions of the form Eq. 1:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

Let $\varphi : C^2 \to C$ and let h be univalent in UIf p is analytic in U and satisfies the differential subordination $\varphi(p(z)), zp'(z)) \prec h(z)$ then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, $p \prec q$. If p and $\varphi(p(z)), zp'(z))$ are univalent in U and satisfy the differential superordination $h(z) \prec \varphi(p(z)), zp'(z))$ then p is called a solution of the differential superordination. An analytic function q is called subordinant of the solution of the differential superordination if $q \prec p$.

Definition 1 (Miller and Mocanu 2003): Denote by Q the set of all functions f(z) that are analytic and injective on $\overline{U} - E(f)$ where $E(f) := \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$ and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U - E(f)$.

MATERIAL AND METHODS

Basically, the method that we will use in this study is via differential subordination given by the famous mathematicians, Miller and Mocanu. We now state the lemmas needed to prove our results.

Lemma 1 (Miller and Mocanu 2003): Let h be convex in U with $h(0) = a, \gamma \neq 0$ with $\Re\{\gamma\} \geq 0$, and $p \in H[a,n] \cap Q$. If $p(z) + \frac{zp'(z)}{\gamma} \lim_{\delta x \to 0}$ is univalent in U and Eq. 2:

$$h(z) \prec p(z) + \frac{zp'(z)}{\gamma} \tag{2}$$

then $q(z) \prec p(z)$, where:

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t) t^{\frac{\gamma}{n} - 1} dt.$$

The function q is convex and is the best subordinant.

Lemma 2 (Miller and Mocanu, 2003): Let q be convex in U and let h be defined by Eq. 3:

$$h(z) = q(z) + \frac{zq'(z)}{\gamma} \tag{3}$$

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with $\Re\{\gamma\} \geq 0$. If $p \in H[a,n] \cap Q$ and $p(z) + \frac{zp'(z)}{\gamma}$ is univalent in U and $q(z) + \frac{zq'(z)}{\gamma} \prec p(z) + \frac{zp'(z)}{\gamma}$, $z \in U$, then $q(z) \prec p(z)$, where:

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t) t^{\frac{\gamma}{n}-1} dt.$$

The function q is the best subordinant.

We will also use the following operator which was defined and studied by the authors see Eq. 4 (Darus and Ibrahim, 2009):

$$\begin{aligned} D^0 f(z) &= f(z) \\ &= z + \sum_{n=2}^{\infty} a_n z^n, \\ D^k_{\alpha,\beta,\lambda} f(z) &= D^1_{\alpha,\beta,\lambda}(D^{k-1}_{\alpha,\beta,\lambda} f(z)) \\ &= z + \sum_{n=2}^{\infty} [\beta(n-1)(\lambda-\alpha)+1]^k a_n z^n \end{aligned} \tag{4}$$

for $\alpha \geq 0, \beta > 0, \lambda > 0, \alpha \neq \lambda$ and $k \in N_0 = N \cup \{0\}$ with $D^k_{\alpha,\beta,\lambda} f(0) = 0$

Remark 1:

(i) When $\alpha = 0$, $\beta = 1$ we receive Al-Oboudi's differential operator see (Al-Oboudi, 2004).
(ii) And when $\alpha = 0$, $\beta = 1$ and $\lambda = 1$ we get Salageans differential operator see (Salagean, 1983).

RESULTS AND DISCUSSION

We shall state our first result as the following.

Theorem 1: Let $h \in H$ be convex in U with $h(0) = 1$. Let $f \in A$ and suppose that $[D^{k+1}_{\alpha,\beta,\lambda} f(z)]'$ is univalent and $[D^k_{\alpha,\beta,\lambda} f(z)]' \in H[1,n] \cap Q$. If Eq. 5 and 6:

$$h(z) \prec [D^{k+1}_{\alpha,\beta,\lambda} f(z)]' \tag{5}$$

Then:

$$q(z) \prec [D^k_{\alpha,\beta,\lambda} f(z)]' \tag{6}$$

Where:

$$q(z) = \frac{1}{\beta(\lambda-\alpha) n z^{\frac{1}{\beta(\lambda-\alpha)n}}} \int_0^z h(t) t^{\frac{1}{\beta(\lambda-\alpha)n}-1} dt.$$

The function q is convex and is the best subordinant.

Proof: By using the properties of the operator $D^k_{\alpha,\beta,\lambda} f(z)$ we have Eq. 7:

$$\begin{aligned} D^{k+1}_{\alpha,\beta,\lambda} f(z) &= [1-\beta(\lambda-\alpha)] D^k_{\alpha,\beta,\lambda} f(z) \\ &+ \beta(\lambda-\alpha) z [D^k_{\alpha,\beta,\lambda} f(z)]'(z). \end{aligned} \tag{7}$$

Differentiating (7), we have Eq. 8

$$\begin{aligned} D^{k+1}_{\alpha,\beta,\lambda} f'(z) &= D^k_{\alpha,\beta,\lambda} f(z) \\ &+ \beta(\lambda-\alpha) z D^k_{\alpha,\beta,\lambda} f(z)''(z). \end{aligned} \tag{8}$$

Consider (8):

$$p(z) = [D^k_{\alpha,\beta,\lambda} f(z)]', \ (z \in U) \frac{\partial^2 \Omega}{\partial u \partial v}$$

then (8) becomes Eq. 9:

$$[D^{k+1}_{\alpha,\beta,\lambda} f(z)]' = p(z) + \beta(\lambda-\alpha) z p'(z) \tag{9}$$

Assume that $\gamma = \frac{1}{\beta(\lambda-\alpha)}$ in Lemma 1, we obtain $q(z) \prec p(z) = [D^k_{\alpha,\beta,\lambda} f(z)]'$.

Where:

$$q(z) = \frac{1}{\beta(\lambda-\alpha) n z^{\frac{1}{\beta(\lambda-\alpha)n}}} \int_0^z h(t) t^{\frac{1}{\beta(\lambda-\alpha)n}-1} dt.$$

The function q is convex and is the best subordinant.

When $\alpha = 0$, $\beta = 1$, we have the next result which can be found in (Catas, 2009).

Corollary 1: Let $h \in H$ be convex in U with $h(0) = 1$. Let $f \in A$ and suppose that $[D^{k+1}_{0,1,\lambda} f(z)]'$ is univalent and $[D^k_{0,1,\lambda} f(z)]' \in H[1,n] \cap Q$. If Eq. 10 and 11:

$$h(z) \prec [D^{k+1}_{0,1,\lambda} f(z)]' \tag{10}$$

Then:

$$q(z) \prec [D^k_{0,1,\lambda} f(z)]' \tag{11}$$

Where:

$$q(z)=\frac{1}{\lambda n z^{\frac{1}{\lambda n}}}\int_0^z h(t)t^{\frac{1}{\lambda n}-1}dt.$$

The function q is convex and is the best subordinant.

Theorem 2: Let $h\in H$ be convex in U with h(0)=1. Let $f\in A$ and suppose that $[D^k_{\alpha,\beta,\lambda}f(z)]'$ is univalent and $\frac{D^k_{\alpha,\beta,\lambda}f(z)}{z}\in H[1,n]\cap Q$. If Eq. 12 and 13:

$$h(z)\prec[D^k_{\alpha,\beta,\lambda}f(z)]' \tag{12}$$

Then:

$$q(z)\prec\frac{D^k_{\alpha,\beta,\lambda}f(z)}{z},\ (z\neq 0) \tag{13}$$

Where:

$$q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt.$$

The function q is convex and is the best subordinant.

Proof: By setting:

$$p(z):=\frac{D^k_{\alpha,\beta,\lambda}f(z)}{z},\ (z\in U)$$

Or Eq. 14:

$$D^k_{\alpha,\beta,\lambda}f(z)=zp(z),\ (z\in U). \tag{14}$$

Differentiating (14), we have Eq. 15:

$$[D^k_{\alpha,\beta,\lambda}f(z)]'=p(z)+zp'(z). \tag{15}$$

By using Lemma 1, with $\gamma=1$ we get:

$$q(z)\prec p(z)=\frac{D^k_{\alpha,\beta,\lambda}f(z)}{z},\ (z\in U)$$

Where:

$$q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt.$$

The function q is convex and is the best subordinant.

Theorem 3: Let q be convex and let h be defined by:

$$h(z)=q(z)+\beta(\lambda-\alpha)zq'(z),\ (z\in U)$$

Let $f\in A$ and suppose that $[D^{k+1}_{\alpha,\beta,\lambda}f(z)]'$ is univalent and $[D^k_{\alpha,\beta,\lambda}f(z)]'\in H[1,n]\cap Q$. If Eq. 16 and 17:

$$h(z)\prec[D^{k+1}_{\alpha,\beta,\lambda}f(z)]' \tag{16}$$

Then:

$$q(z)\prec[D^k_{\alpha,\beta,\lambda}f(z)]' \tag{17}$$

Where:

$$q(z)=\frac{1}{\beta(\lambda-\alpha)nz^{\frac{1}{\beta(\lambda-\alpha)n}}}\int_0^z h(t)t^{\frac{1}{\beta(\lambda-\alpha)n}-1}dt.\lim_{\delta x\to 0}$$

The function q is convex and is the best subordinant.

Proof: By using the properties of the operator Eq. 18:

$$[D^{k+1}_{\alpha,\beta,\lambda}f(z)]'=[D^k_{\alpha,\beta,\lambda}f(z)]' +\beta(\lambda-\alpha)z[D^k_{\alpha,\beta,\lambda}f(z)]''(z). \tag{18}$$

By denoting:

$$p(z):=[D^k_{\alpha,\beta,\lambda}f(z)]',\ (z\in U)$$

then (18) becomes Eq. 19:

$$[D^{k+1}_{\alpha,\beta,\lambda}f(z)]'=p(z)+\beta(\lambda-\alpha)zp'(z). \tag{19}$$

Assume that $\gamma=\frac{1}{\beta(\lambda-\alpha)}$ in Lemma 2, we obtain:

$$q(z)\prec p(z)=[D^k_{\alpha,\beta,\lambda}f(z)]'$$

Where:

$$q(z)=\frac{1}{\beta(\lambda-\alpha)nz^{\frac{1}{\beta(\lambda-\alpha)n}}}\int_0^z h(t)t^{\frac{1}{\beta(\lambda-\alpha)n}-1}dt.$$

The function q is convex and is the best subordinant.

When $\alpha = 0$, $\beta = 1$, we have the next result which can be found in (Catas, 2009).

Corollary 2: Let q be convex and let h be defined by:

$$h(z) = q(z) + \lambda zq'(z), (z \in U)$$

Let $f \in A$ and suppose that $[D^{k+1}_{0,1,\lambda} f(z)]'\partial$ is univalent and $[D^{k}_{0,1,\lambda} f(z)]' \in H[1,n] \cap Q$. If Eq. 20 and 21:

$$h(z) \prec [D^{k+1}_{0,1,\lambda} f(z)]' \quad (20)$$

Then:

$$q(z) \prec [D^{k}_{0,1,\lambda} f(z)]' \quad (21)$$

Where:

$$q(z) = \frac{1}{\lambda n z^{\frac{1}{\lambda n}}} \int_0^z h(t) t^{\frac{1}{\lambda n} - 1} dt.$$

The function q is convex and is the best subordinant.

Theorem 4: Let q be convex and let h be defined by:

$$h(z) = q(z) + zq'(z), (z \in U)$$

Let $f \in A$ and suppose that $[D^{k}_{\alpha,\beta,\lambda} f(z)]'$ is univalent and $\frac{D^{k}_{\alpha,\beta,\lambda} f(z)}{z} \in H[1,n] \cap Q$. If Eq. 22 and 23:

$$h(z) \prec [D^{k}_{\alpha,\beta,\lambda} f(z)]' \quad (22)$$

Then:

$$q(z) \prec \frac{D^{k}_{\alpha,\beta,\lambda} f(z)}{z} \quad (23)$$

Where:

$$q(z) = \frac{1}{n z^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n} - 1} dt.$$

The function q is convex and is the best subordinant.

Proof: By setting:

$$p(z) := \frac{D^{k}_{\alpha,\beta,\lambda} f(z)}{z}, (z \in U)$$

Or Eq. 24:

$$D^{k}_{\alpha,\beta,\lambda} f(z) = zp(z), (z \in U). \quad (24)$$

Differentiating (24), we have:

$$[D^{k}_{\alpha,\beta,\lambda} f(z)]' = p(z) + zp'(z).$$

Note: Some other study can also be found in the articles written by (Al-Shaqsi *et al*., 2010). Most of the results may look similar, but they do have plentiful applications in area of studies.

CONCLUSION

We can see here that by creating new differential operator, many other results can be solved. In fact, classical results such as the distortion, radii of star likeness and convexity can also be obtained readily by using suitable methods.

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