Journal of Mathematics and Statistics 8 (4): 506-516, 2012
ISSN 1549-3644

doi:10.3844/jmssp.2012.506.516 Published Online 8 (4) 2012 (http://www.thescipub.com/jmss.toc)

Limit Theorems for a Class of Additive Functionals of Symmetric Stable Process and Fractional Brownian Motion in Besov-Orlicz Spaces

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Received 2012-10-20, Revised 2012-11-28; Accepted 2012-11-28

ABSTRACT

We use the Potter's Theorem and the tightness criterion in Besov-Orlicz spaces, recently proved, to generalize some limit theorem for occupation times problem of certain self-similar process, namely symmetric stable process of index 1<α≤ and fractional Brownian motion of Hurst parameter 0<H<1. We give also strong approximation version of our limit theorem, more precisely, we show L^p-estimate version.

Keywords: Besov-Orlicz Spaces, Limit Theorems, Strong Approximation, Selfsimilar Process, Fractional Derivative, Additive Functional, Regularly Varying Function

1. INTRODUCTION

In the present study, we are interested in the limit theorems of a class of continuous additive functionals of some self-similar process, namely stable process and fractional Brownian motion. The interesting properties such as self similarity and stationarity of increments make these processes good candidates as models for different phenomena, related to financial mathematics and telecommunications.

Most of the estimates in this paper contain unspecified positive constants. We use the same symbol C for these constants, even when they vary from one line to the next. We first collect some facts about these processes.

Let $X^\alpha = \{X_t^\alpha ; t \geq 0\}$ be a real valued symmetric stable process of index $1<\alpha\leq 2$, with $X_0^\alpha = 0$, (α-SSP for brevity). The sample paths of X_t^α are right-continuous with left limits a.s. (cadlag for brevity) and has stationary independent increments with characteristic function:

$$E\exp(i\lambda X_t^\alpha) = \exp(-t|\lambda|^\alpha),\ \forall t \geq 0,\ \lambda \in \mathbb{R}$$

It is known from Boylan (1964) and Barlow (1988) that X^α admits a continuous local time process $\{L(t, x); t\geq 0, x\in \mathbb{R}\}$ satisfying the scaling property:

$$\left\{L(\lambda t, x\lambda^{\frac{1}{\alpha}})\right\}_{t\geq 0} \underline{\underline{L}} \left\{\lambda^{\frac{\alpha-1}{\alpha}} L(t,x)\right\}_{t\geq 0},\ \forall \lambda > 0$$

where, "$\underline{\underline{L}}$" means the equality in the sense of the finite-dimensional distributions and the occupation density formula:

$$\int_0^t f(X_s^\alpha)ds = \int_{\mathbb{R}} f(x)L(t,x)dx$$

for any bounded or nonnegative Borel function f.

Moreover, in Marcus and Rosen (1992) and in Ait Ouahra and Eddahbi (2001), for each T > 0 fixed, there

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exists a constant $0 < C < \infty$ such that for any integer $p \geq 1$, all $0 \leq t, s \leq T$ and all x; $y \in \mathbb{R}$:

$$\| L(t,x) - L(s,x) \|_{2p} \leq C((2p)!)^{\frac{1}{2p}} |t-s|^{\frac{\alpha-1}{\alpha}},$$

$$\|L(t,x) - L(t,y)\|_{2p} \leq C((2p)!)^{\frac{1}{2p}} |x-y|^{\frac{\alpha-1}{2}},$$

$$\|L(t,x) - L(s,x) - L(t,y) + L(s,y)\|_{2p} \leq C((2p)!)^{\frac{1}{2p}} |t-s|^{\frac{\alpha-1}{2\alpha}} |x-y|^{\frac{\alpha-1}{2}}$$

Where:

$$\|.\|_{2p} = (E)|.|^{2p})^{\frac{1}{2p}}$$

Given a constant $H \in]0, 1[$, the fractional Brownian motion (fBm for brevity) with Hurst parameter H is the real valued centered Gaussian process $B^H = \{B_t^H; t \geq 0\}$ with stationary increments and covariance function:

$$R(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$$

However, the increments of fBm are not independent except in the Brownian motion case ($H = \frac{1}{2}$, Bm for brevity). The dependence structure of the increments is modeled by a parameter H. fBm is self-similar with exponent τ = H and his local time satisfies the occupation density formula and the scaling property.

Notice that the α-SSP is self-similar with exponent $\tau = \frac{1}{\alpha}$.

Geman and Horowitz (1980) proved that the local time of fBm exists and has a.s. Holder continuous modification of order $\gamma_0 - \varepsilon$ in space and of order $1-H-\varepsilon$ in time for any $\varepsilon > 0$ and $\gamma_0 = \min\left(1, \frac{1-H}{2H}\right)$. More precisely, it is proved by Xiao (1997), that for each $T > 0$ fixed, there exists a constant $0 < C < \infty$ such that for any integer $p \geq 1$, all $0 \leq t, s \leq T$ and all x, $y \in \mathbb{R}$:

$$\|L(t,x) - L(s,x)\|_{2p} \leq C((2p)!)^{\frac{H}{2p}} |t-s|^{1-H},$$

$$\|L(t,x) - L(t,y)\|_{2p} \leq C((2p)!)^{\frac{2\delta + H(1+\delta)}{2p}} |x-y|^{\delta},$$

$$\|L(t,x) - L(s,x) - L(t,y) + L(s,y)\|_{2p}$$

$$\leq C((2p)!)^{\frac{2\delta + H(1+\delta)}{2p}} |t-s|^{1-H(1+\delta)} |x-y|^{\delta} \text{ for any } 0 < \delta < \gamma_0$$

Notice that $0 < \frac{1-H}{2+H} < \frac{1-H}{2} < \gamma_0$ and for any $0 < \delta < \frac{1-H}{2+H}$, we have:

$$2\delta + H(1+\delta) < 1$$

Therefore the last regularities becomes:

$$\|L(t,x) - L(s,x)\|_{2p} \leq C((2p)!)^{\frac{1}{2P}} |t-s|^{1-H},$$

$$\|L(t,x) - L(t,y)\|_{2p} \leq C((2p)!)^{\frac{1}{2p}} |x-y|^{\delta},$$

$$\|L(t,x) - L(s,x) - L(t,y) + L(s,y)\|_{2p} \leq C((2p)!)^{\frac{1}{2p}} |t-s|^{1-H(1+\delta)} |x-y|^{\delta} \text{ for any } 0 < \delta < \frac{1-H}{2+H} < \gamma_0$$

For an excellent summary of fBm, the reader is referred to Mandelbrot and Ness (1968) and Samorodnitsky and Taqqu (1994).

Remark 1

- Notice that for $H = \frac{1}{2}$ (respectively α = 2), B^H (respectively X^α) is a Bm
- The α-SSP has independent increments, contrary to fBm which does not have independent increments, except for the special case of the Bm
- B^H has a.s. Holder continuous modification of order $\beta < H$ but X^α is just cadlag

Throughout this study, we use the same symbol $Y^\tau = \{Y_t^\tau, t \geq 0\}$ to denote α-SSP $\left(\tau = \frac{1}{\alpha}\right)$ or fBm (τ = H) and we denote $\{L(t, x); t \geq 0, x \in R\}$ its local time. Then, for each $T > 0$ fixed, there exists a constant $0 < C < \infty$ such that for any integer $p \geq 1$, all $0 \leq t, s \leq T$ and all x, $y \in \mathbb{R}$ Equation 1-3:

$$\|L(t,x) - L(s,x)\|_{2p} \leq C((2p)!)^{\frac{1}{2p}} |t-s|^{1-\tau} \tag{1}$$

$$\|L(t,x)-L(t,y)\|_{2p} \le C((2p)!)^{\frac{1}{2p}}|x-y|^{\delta} \quad (2)$$

$$\begin{aligned}&\|L(t,x)-L(s,x)-L(t,y)+L(s,y)\|2p\\&\le C((2p)!)^{\frac{1}{2p}}|t-s|^{1-\tau(1+\delta)}|x-y|^{\delta}\end{aligned} \quad (3)$$

where, $\delta=\delta_0=\frac{1-\tau}{2\tau}$ for α-SSP and $0<\delta<\delta_0=\frac{1-\tau}{2+\tau}<\gamma_0$ for fBm.

Self-similar process arise naturally in limit theorems of random walks and other stochastic process. Many authors have studied the limit theorems of the process Equation 4:

$$\frac{1}{\lambda^{1-\tau(1+\gamma)}}\int_0^{\lambda t} f(Y_s^{\tau})ds \quad (4)$$

where, $f=D_{\pm}^{\gamma}g$ and $g\in C^{\beta}\cap L^1(\mathbb{R})$ with compact support. We cite Yamada (1986; 1996) for Bm $\left(\tau=\frac{1}{2}\right)$, Shieh (1996) for fBm ($\tau$ = H) and Fitzsimmons and Getoor (1992) for α-SSP $\left(\tau=\frac{1}{\alpha}\right)$. All these results are established in the space of continuous functions. Ouahra and Eddahbi (2001) extended the results of Fitzsimmons and Getoor (1992) to Holder spaces and Ouahra *et al.* (2002) in Besov spaces and recently, Ouahra *et al.* (2011) in Besov-Orlicz spaces. The result of Shieh (1996) was extended by Ouahra and Ouali (2009) in Besov spaces.

The objective of the present study is to study in Besov-Orlicz spaces, the limit theorem of the process (4), where f has the form $f=K_{\pm}^{l,\gamma}g$, (see the definition of $K_{\pm}^{l,\gamma}$ below).

We recall the following definition which will be useful in the sequel.

Definition 1

A measurable function $U:\mathbb{R}^+\to\mathbb{R}^+$ is regularly varying at infinity in (Karamata's sense), with a real exponent r, if for all t positive:

$$\lim_{x\to+\infty}\frac{U(tx)}{U(x)}=t^r$$

If r = 0, we call U slowly varying function denoted by l. We see that $U(x)=x^r l(x)$.

We are interested in the behavior of l at + ∞, then we can assume for example that l is bounded on each interval of the form [0, a]; (a > 0).

In what follows, we assume that for $\gamma>0$, k_γ is a regularly varying function with exponent $-(1+\gamma)$ defined by:

$$k_\gamma(y)=\begin{cases}\frac{l(y)}{y^{1+\gamma}} & \text{if } y>0\\ 0 & \text{if } y\le 0\end{cases}$$

where, l is slowly varying function at +∞, continuously differentiable and $l(x)>0$ for all $x>0$ and $l(0^+)=1$, (Bingham *et al.* (1989), Theorem 1.3.3).

For any $\gamma\in\,]0,\beta[$ and $g\in C^{\beta}\cap L^1(\mathbb{R})$, we define:

$$k_{\pm}^{l,\gamma}g(x):=\frac{1}{\Gamma(-\gamma)}\int_0^{+\infty}k_\gamma(y)[g(x\pm y)-g(x)]dy$$

and we put:

$$K^{l,\gamma}:=K_{+}^{l,\gamma}-K_{-}^{l,\gamma}$$

$K^{l,\gamma}$ is called the generalized fractional derivative.

Remark 2

By (2) and the occupation time formula we have $L(t,.)\in C^{\beta}\cap L^1(\mathbb{R})$ for some $\beta>0$, then we can define $K^{l,\gamma}L(t,.)(x)$ for any $0<\gamma<\beta$.

The following theorem called Potter's Theorem has played a central role in the proof of our results, (Bingham *et al.*, 1989).

Theorem 1

- If l is slowly varying function, then for any chosen constants A > 1 and $\xi>0$, there exists X = X(A, ξ) such that:

$$\frac{l(y)}{l(x)}\le A\max\left\{\left(\frac{y}{x}\right)^{\xi},\left(\frac{y}{x}\right)^{-\xi}\right\}(x\ge X, y\ge X)$$

- If further, l is bounded away from 0 and ∞ on every compact subset of [0,+∞[, then for every ξ>0, there exists A' = A'(ξ) >1 such that:

$$\frac{l(y)}{l(x)}\le A'\max\left\{\left(\frac{y}{x}\right)^{\xi},\left(\frac{y}{x}\right)^{-\xi}\right\}(x>0, y>0)$$

- If U is regularly varying function with exponent $r \in \mathbb{R}$, then for any chosen $A > 1$ and $\xi > 0$, there exists $X = X(A, \xi)$ such that:

$$\frac{U(y)}{U(x)} \le A \max\left\{\left(\frac{y}{x}\right)^{r+\xi}, \left(\frac{y}{x}\right)^{r-\xi}\right\} (x \ge X, y \ge X)$$

The monograph by Seneta (1976) contains a very readable exposition of the basic theory of regularly varying functions on $\mathbb{R}$.

The following proposition is the main result of this section. It is a consequence of a simple computation integral.

Proposition 1

For $h: \mathbb{R} \to \mathbb{R}$ and $a > 0$, we denote by h_a the function $x \to h(ax)$, then Equation 5:

$$K_{\pm}^{l,\gamma}(h_a) = a^{\gamma}\left(K_{\pm}^{l\left(\frac{\cdot}{a}\right),\gamma}\right)a, \forall\gamma > 0, \forall a > 0 \tag{5}$$

Remark 3

- $K_{+}^{l,\gamma}$ and $K_{-}^{l,\gamma}$ satisfy the switching identity Equation 6:

$$\int_{\mathbb{R}} f(x)K_{-}^{l,\gamma}g(x)dx = \int_{\mathbb{R}} g(x)K_{+}^{l,\gamma}f(x)dx \tag{6}$$

for any $f, g \in C^{\beta} \cap L^1(\mathbb{R})$ and $\gamma \in]0, \beta[$.

- If we take $l \equiv 1$, we recover the definition of fractional derivative, Yamada (1985) and Samko *et al*. (1993) and the references therein.

The remainder of this study is organized as follows: we present some basic facts about Besov-Orlicz spaces. We give the proof of our main result. Finally, we state and prove strong approximation version of our limit theorem.

1.1. The Functional Framework

We will present a brief survey of Besov-Orlicz spaces. For more details, we refer the reader to Boufoussi (1994) and Ciesielski *et al*. (1993).

Let (Ω, Σ, μ) be a σ-finite measure space. We denote by $L^p(\Omega)$, $1 \le p < +\infty$, the space of Lebesgue integrable real valued functions f on Ω with exponent p endowed with the norm:

$$\|f\|_p = \left(\int_{\Omega} |f(.)|^p d\mu(.)\right)^{\frac{1}{p}}$$

The Orlicz space $L_{M\beta(d\mu)}(\Omega)$ corresponding to the Young function $M_\beta(x) = e^{|x|\beta} - 1$, $\beta \ge 1$ is the Banach space of real valued measurable functions f on Ω endowed with the norm:

$$\|f\|_{M_\beta}(d\mu) = \inf_{\lambda>0}\left\{\int_{\Omega} M_\beta\left(\left|\frac{f(.)}{\lambda}\right|\right)d\mu(.) < 1\right\}$$

This norm is equivalent to the norm of Luxemburg (1955) given by:

$$\|f\|_{M_\beta}^{*}(d\mu) = \inf_{\lambda>0}\frac{1}{\lambda}\left\{1 + \int_{\Omega} M_\beta(|\lambda f(.)|)d\mu(.)\right\}$$

In case of (Ω, Σ, P) being a probability space, the Orlicz norm become:

$$\|f\|_{M_\beta}(dP) = \inf_{\lambda>0}\left\{E\left(M_\beta\left(\left|\frac{f}{\lambda}\right|\right)\right) < 1\right\}$$

In this study, we use the following equivalence norm in $L_{M_\beta(d\mu)}(\Omega)$, (see for example Ciesielski *et al*. (1993):

$$\|f\|_{M_\beta}(d\mu) \sim \sup_{p\ge1}\frac{\|f\|_p}{p^{\frac{1}{\beta}}}$$

Benchekroun and Benkirane (1995) have proved that for any open $A \subset \Omega$ and any $f \in L_{M_\beta(d\mu)}(A)$, we have Equation 7:

$$\|f.g\|_{M_\beta(d\mu)} \le \|g\|\infty\|f\|_{M_\beta(d\mu)} \tag{7}$$

where, $\|g\|_\infty = \sup_{x\in A}|g(x)|$.

These last two results and the Potter's Theorem have played a central role in the proof of our limit theorem.

The modulus of continuity of a Borel function f: [0, 1] $\to \mathbb{R}$ in Orlicz norm is defined by:

$$\omega_{M_\beta}(f,t) = \sup_{0\le h\le t}\|\Delta_h f\|_{M_\beta(dx)}$$

Where:

$$\Delta_h f(t) = 1_{[0,1-h]}(t)[f(t+h) - f(t)]$$

The Besov-Orlicz space, denoted by $B^{\omega_{\mu,v}}_{M_\beta,\infty}$ is a non separable Banach space of real valued continuous functions f on [0, 1] endowed with the norm:

$$\|f\|^{\omega_{\mu,v}}_{M_\beta,\infty} = \|f\|_{M_\beta(dx)} + \sup_{0<t\le 1} \frac{\omega_{M_\beta}(f,t)}{\omega_{\mu,v}(t)}$$

Where:

$$\omega_{\mu,v}(t) = t^\mu \left(1 + \log\left(\frac{1}{t}\right)\right)^v$$

for any $0 < \mu < 1$ and $v > 0$.

Let $\{\varphi_n = \varphi_{j,k},\ j \ge 0,\ k = 1,\dots 2^j\}$ be the Schauder basis. The decomposition and the coefficients of continuous functions f on [0, 1] in this basis are respectively given as follows:

$$f(t) = \sum_{n=0}^{\infty} C_n(f)\varphi_n(t)$$

and:

$$\begin{cases} C_0(f) = f(0), C_1(f) = f(1) - f(0) \\ n = 2^j + k, j \ge 0, k = 1,\dots,2^j \\ C_n(f) = f_{j,k} = 2^{\frac{j}{2}}\left(2f\left(\frac{2k-1}{2^{j+1}}\right) - f\left(\frac{2k-2}{2^{j+1}}\right) - f\left(\frac{2k}{2^{j+1}}\right)\right) \end{cases}$$

We consider the separable Banach subspace of $B^{\omega_{\mu,v}}_{M_\beta,\infty}$ defined as follows:

$$B^{\omega_{\mu,v},0}_{M_\beta,\infty} = \{f \in B^{\omega_{\mu,v}}_{M_\beta,\infty} \,/\, \omega_{M_\beta}(f,t) = 0(\omega_{\mu,v}(t))(t \downarrow 0)\}$$

It is known from Ciesielski *et al*. (1993) that the subspace $B^{\omega_{\mu,v},0}_{M_\beta,\infty}$ corresponds to sequences $(f_{j,k})_{j,k}$ such that:

$$\lim_{j\to+\infty} \frac{2^{-j\left(\frac{1}{2}-\mu+\frac{1}{p}\right)}}{p^{\frac{1}{\beta}}(1+j)^v} \left[\sum_{k=1}^{2^j} |f_{j,k}|^p\right]^{\frac{1}{p}} = 0$$

For the proof of our results, we need the following tightness criterion in the subspace $B^{\omega_{\mu,v},0}_{M_\beta,\infty}$ (Ouahra *et al*., 2011).

Theorem 2

Let $\{X^n_t : t \in [0,1]\}_{n\ge 1}$ be a sequence of stochastic processes satisfying:

- $X^n_0 = 0$ for all $n \ge 1$
- There exists a constant $0 < C < \infty$ such that for any $(t, s) \in [0.1]^2$:

$$\|X^n_t - X^n_s\|_{M_\beta(dP)} \le C|t-s|^\mu$$

where, $0 < \mu < 1$. Then, the sequence $\{X^n_t : t \in [0,1]\}_{n\ge 1}$ is tight in the space $B^{\omega_{\mu,v},0}_{M_\beta,\infty}$, for all $v > 1$ and $\beta \ge 1$.

We end this section by the following regularity of local time.

Corollary 1

For each $T > 0$ fixed, there exists a constant $0 < C < \infty$ such that for all $0 \le t, s \le T$ and all $x \in \mathbb{R}$:

$$\|L(t,x) - L(s,x)\|_{M_1(dP)} \le C|t-s|^{1-\tau}$$

Proof

By virtue of the equivalence norm in $L_{M_1(d\mu)}(\Omega)$ and (1), there exists a constant $0 < C < \infty$, such that:

$$\begin{aligned} \|L(t,x) - L(s,x)\|_{M_1(dP)} &\le C\sup_{p\ge 1} \frac{\|L(t,x) - L(s,x)\|_{2p}}{2p} \\ &\le C\sup_{p\ge 1} \frac{((2p)!)^{\frac{1}{2p}}}{2p}|t-s|^{1-\tau} \\ &\le C|t-s|^{1-\tau} \end{aligned}$$

where we have used in the last inequality the fact that $((2p)!)^{\frac{1}{2p}} \le 2p$.

This complete the proof of Corollary 1.

1.2. Limit Theorems

In order to establish our limit theorem, we need the following regularities.

Lemma 1

Let T > 0 fixed, $0 < \gamma < \delta$ and $K \in \{K_{\pm}^{l,\gamma}, K^{l,\gamma}\}$. There exists a constant $0 < C < \infty$ such that for all $0 \le t, s \le T$, all $x \in \mathbb{R}$ and any integer $p \ge 1$:

$$\|KL(t,.)(x) - KL(s,.)(x)\|_{2p} \le C((2p)!)^{\frac{1}{2p}} |t-s|^{1-\tau(1+\gamma)}$$

Remark 4

This regularity is similar to that given in Ouahra and Eddahbi (2001) for fractional derivatives of local time of α-SSP and in Ouahra and Ouali (2009) for fBm case.

Proof of Lemma 1

We treat only the case $K = K_{+}^{l,\gamma}$, the other cases are similar. Let b = $|t-s|^{\tau}$. By the definition of $K_{+}^{l,\gamma}$ we have:

$$\|K_{+}^{l,\gamma}L(t,.)(x) - K_{+}^{l,\gamma}L(s,.)(x)\|_{2p}$$
$$\le \frac{1}{|\Gamma(-\gamma)|}\int_0^b l(u)\frac{\|L(t,x+u) - L(s,x+u) - L(t,x) + L(s,x)\|_{2p}}{u^{1+\gamma}}du$$
$$+\frac{1}{|\Gamma(-\gamma)|}\int_b^{+\infty} l(u)\frac{\|L(t,x+u) - L(s,x+u) - L(t,x) + L(s,x)\|_{2p}}{u^{1+\gamma}}du$$
$$:= I_1 + I_2$$

We estimate I_1 and I_2 separately.

Estimate of I_1.

Since l is bounded on each compact in $\mathbb{R}^+$, it follows from (3) that:

$$I_1 \le C((2p)!)^{\frac{1}{2p}} |t-s|^{1-\tau(1+\delta)} b^{\delta-\gamma}$$
$$\le C((2p)!)^{\frac{1}{2p}} |t-s|^{1-\tau(1+\gamma)}$$

Now we return to estimate I_2.

Potter's Theorem with $0 < \xi < \gamma$ implies the existence of $A(\xi) > 1$ such that:

$$l(u) \le A(\xi) l(b) \left(\frac{u}{b}\right)^{\xi}$$

Combining this fact with (1), we obtain:

$$I_2 \le C((2p)!)^{\frac{1}{2p}} |t-s|^{1-\tau(1+\gamma)}$$

The proof of Lemma 1 is done.

We prove, in the same way as before the following result. It will be useful to prove the tightness in Theorem 3.

Corollary 2

Let T>0 fixed and $0 < \gamma < \delta$. There exists a constant $0 < C < \propto$ such that for all $0 \le t, s \le T$, all $x \in \mathbb{R}$ and n large enough:

$$[l(n^{\tau})]^{-1} \left\| K_{\pm}^{l(\frac{\cdot}{n-\tau}),\gamma} L(t,.)\left(\frac{x}{n^{\tau}}\right) - K_{\pm}^{l(\frac{\cdot}{n-\tau}),\gamma} L(s,.)\left(\frac{x}{n^{\tau}}\right) \right\|_{2p}$$
$$\le C((2p)!)^{\frac{1}{2p}} |t-s|^{1-\tau(1+\gamma)}$$

Proof

We treat only the case $K_{+}^{l(\frac{\cdot}{n-\tau}),\gamma}$ the other cases are similar.

Let $b = |t-s|^{\tau}$. By the definition of $K_{+}^{l(\frac{\cdot}{n-\tau}),\gamma}$, we have:

$$[l(n^{\tau})^{-1}] \left\| K_{+}^{l\left(\frac{\cdot}{n-\tau}\right)\gamma} L(t,.)\left(\frac{x}{n^{\tau}}\right) - K_{+}^{l\left(\frac{\cdot}{n-\tau}\right),\gamma} L(s,.)\left(\frac{x}{n^{\tau}}\right) \right\|_{2p}$$
$$\le \frac{1}{|\Gamma(-\gamma)|}\int_0^b \frac{l(n^{\tau}\mu)}{(n^{\tau})}$$
$$\frac{\left\| L\left(t, \frac{x}{n^{\tau}} + u\right) - L\left(s, \frac{x}{n^{\tau}} + u\right) - L\left(s, \frac{x}{n^{\tau}}\right) + L\left(s, \frac{x}{n^{\tau}}\right) \right\|_{2p}}{u^{1+\gamma}} du$$
$$+\frac{1}{|\Gamma(-\gamma)|}\int_b^{+\infty} \frac{l(n^{\tau}u)}{l(n^{\tau})}$$
$$\frac{\left\| L\left(t, \frac{x}{n^{\tau}} + u\right) - L\left(s, \frac{x}{n^{\tau}} + u\right) - L\left(t, \frac{x}{n^{\tau}}\right) + L\left(s, \frac{x}{n^{\tau}}\right) \right\|_{2p}}{u^{1+\gamma}} du$$
$$:= J_1 + J_2$$

We estimate J_1 and J_2 separately.

Estimate of J_1: It follows from (3) that:

$$J_1 \le C((2p)!)^{\frac{1}{2p}} \sup_{u \in \mathbb{R}+} \frac{l(n^{\tau}u)}{l(n^{\tau})} |t-s|^{1-\tau(1+\delta)} b^{\delta-\gamma}$$
$$\le C((2p)!)^{\frac{1}{2p}} \sup_{u \in \mathbb{R}+} \frac{l(n^{\tau}u)}{l(n^{\tau})} |t-s|^{1-\tau(1+\gamma)}$$

Now we return to estimate J_2.

Potter's Theorem with $0 < \xi < \gamma$ implies the existence of $A(\xi) > 1$ such that:

$$l(n^\tau u) \leq A(\xi) l(n^\tau b)\left(\frac{u}{b}\right)^{\xi}$$

Combining this fact with (1), we obtain:

$$J_2 \leq C((2p)!)^{\frac{1}{2p}} \frac{l(n^\tau b)}{l(n^\tau)} |t-s|^{1-\tau(1+\gamma)}$$

Finally, by using the fact that:

$$\lim_{n\to+\infty} \frac{l(n^\tau u)}{l(n^\tau)} = 1$$

we complete the proof of Corollary 2.

Remark 5

As in Corollary 1, for $0 < \gamma < \delta$, there exists a constant $0 < C < \infty$ such that for all $0 \leq t, s \leq T$, all $x \in \mathbb{R}$ and n large enough:

$$[l(n^\tau)]^{-1} \left\| k_{\pm}^{l\left(\frac{\cdot}{n-\tau}\right),\gamma} L(t,.)\left(\frac{x}{n^\tau}\right) - K_{\pm}^{l\left(\frac{\cdot}{n-\tau}\right),\gamma} L(s,.)\left(\frac{x}{n^\tau}\right) \right\|_{M_1(dP)}$$
$$\leq C |t-s|^{1-\tau(1+\gamma)}$$

Now we are ready to state the main result of this section.

Theorem 3

Let $0 < \gamma < \delta$. Suppose $f = K_{\pm}^{1,\gamma} g$ where $g \in C^\beta \cap L^1(\mathbb{R})$ with compact support for some $\gamma < \beta$. Then as $n \to +\infty$, the sequence of process:

$$\left\{[n^{1-\tau(1+\gamma)} l(n^\tau)]^{-1} \int_0^{nt} f(Y_s^\tau) ds\right\}_{t\geq 0}$$

converges in law to the process:

$$\left\{\left[\int_{\mathbb{R}} g(x)dx\right] D_{\mp}^{\gamma} L(t,.)(0)\right\}_{t\geq 0}$$

in the Besov-Orlicz space $B_{M_1,\infty}^{\omega_{1-\tau}(1+\gamma),v,0}$ for all $v > 1$.

Remark 6

Notice that even if f is not a fractional derivative of some function g, the limiting process is fractional derivative of local time.

Proof

Case of α-SSP. By Fitzsimmons and Getoor (1992), (Remark 3.18), the finite-dimensional distributions of:

$$A_t^n = [n^{1-\tau(1+\gamma)} l(n^\tau)]^{-1} \int_0^{nt} f(Y_s^\tau) ds$$

converge as $n \to +\infty$ to the finite-dimensional distributions of:

$$\left[\int_{\mathbb{R}} g(x)dx\right] D_{\mp}^{\gamma} L(t,.)(0)$$

So to prove this theorem, we need only to show the tightness of the processes A_t^n in the Besov-Orlicz space $B_{M_1,\infty}^{\omega_{1-\tau}(1+\gamma),v\,0}$ for any $v > 1$.

By the occupation density formula and the scaling property of local time, we have:

$$\| A_t^n - A_s^n \|_{M_1(dP)} = \left\| \frac{1}{l(n^\tau) n^{1-\tau(1+\gamma)}} \left(\int_0^{nt} f(Y_u^\tau) du - \int_0^{ns} f(Y_u^\tau) du \right) \right\|_{M_1(dP)}$$
$$= n^{\tau\gamma}[l(n^\tau)^{-1}] \left\| \int_{\mathbb{R}} f(x) L\left(t, \frac{x}{n^\tau}\right) dx - \int_{\mathbb{R}} f(x) L\left(s, \frac{x}{n^\tau}\right) dx \right\|_{M_1(dP)}$$
$$= n^{\gamma\tau}[l(n^\tau)]^{-1} \left\| \int_{\mathbb{R}} k_{+}^{1,\gamma} g(x) \left[L\left(t, \frac{x}{n^\tau}\right) - L\left(s, \frac{x}{n^\tau}\right) dx \right] \right\|_{M_1(dP)}$$
$$= n^{\gamma\tau}[l(n^\tau)]^{-1} \left\| \int_{\mathbb{R}} g(x) [K_{-}^{1,\gamma} L\left(t, \frac{\cdot}{n^\tau}\right)(x) - K_{-}^{1,\gamma} L\left(s, \frac{\cdot}{n^\tau}\right)(x)] dx \right\|_{M_1(dP)}$$

Therefore, it follows from (5) and (7), that:

$$\left\| A_t^n - A_s^n \right\|_{M_1(dp)} \leq C[l(n^\tau)]^{-1}$$
$$\int_s \left\| g(x) \begin{pmatrix} k_{-}^{l\left(\frac{\cdot}{n-\tau}\right),\gamma} L(t,.)\left(\frac{x}{n^\tau}\right) - \\ k_{-}^{l\left(\frac{\cdot}{n-\tau}\right),\gamma} L(s,.)\left(\frac{x}{n^\tau}\right) \end{pmatrix} \right\|_{M_1(dp)} dx$$
$$\leq C \int_S \|g\| \infty [l(n^\tau)]^{-1}$$
$$\left\| \begin{matrix} K_{-}^{l\left(\frac{\cdot}{n-\tau}\right),\gamma} L(t,.)\left(\frac{x}{n^\tau}\right) - K_{-}^{l\left(\frac{\cdot}{n-\tau}\right),\gamma} \\ L(s,.)\left(\frac{x}{n^\tau}\right) - K_{-}^{l\left(\frac{\cdot}{n-\tau}\right),\gamma} L(s,.)\left(\frac{x}{n^\tau}\right) \end{matrix} \right\|_{M_1(dp)} dx$$

where, S = supp(g).

Thanks to Remark 5, for n large enough, we have:

$$\|A_t^n - A_s^n\|_{M_1(dP)} \le C|t-s|^{1-\tau(1+\gamma)}$$

Case of fBm. By analogous arguments using in Fitzsimmons and Getoor (1992), (Remark 3.18), in the case of α-SSP, we obtain the convergence of the finite-dimensional distributions of the processes A_t^n. The tightness follows easily as in the case of α-SSP. This with Theorem 2 completes the proof of Theorem 3.

Remark 7

Our limit theorems in the case of fBm are new even in the space of continuous functions.

1.3. Strong Approximation

We give strong approximation, L^p-estimate, of Theorem 3. Our main result in this paragraph reads.

Theorem 4

Let f be a Borel function on $\mathbb{R}$ satisfying Equation 8:

$$\int_{\mathbb{R}} |x|^k f(x)\,|\,dx < \infty \tag{8}$$

for some k > 0. Then, for any sufficiently small $\varepsilon > 0$ and any integer $p \ge 1$, when t goes to infinity, we have:

$$\left\|\int_0^t k^{1,\gamma} f(Y_s^\tau)ds\right\|_{2p} = \frac{1(f)}{\Gamma(1-\gamma)}\|D^\gamma L(t,.)(0)\|_{2p} + 0(t^{1-\tau(1+\gamma)-\varepsilon})$$

where, I(f) = $\int_{\mathbb{R}} f(x)dx$ and $0 < \gamma < \delta$.

In order to prove Theorem 4, we shall first state and prove some technical lemmas.

Lemma 2

Let $0 < \gamma < \delta$. For any $\varepsilon > 0$ and any integer $p \ge 1$, when t goes to infinity:

$$\sup_{x\in\mathbb{R}}\|K^{1,\gamma}L(t,.)(x)\|_{2p}^{2p} - o(t^{2p(1-\tau(1+\gamma))+\varepsilon})$$

Proof

Using Lemma 1 for s = 0 and the fact that $K^{1,\gamma}(0,.)(x) = 0$ a.s., we get:

$$\sup_{x\in\mathbb{R}}\|K^{1,\gamma}L(t,.)(x)\|_{2p}^{2p} \le Ct^{2p(1-\tau(1+\gamma))}$$

The conclusion follows immediately.

In the same way, using (3) for s = 0 and the fact that L(0, x) = 0 a.s., we get the following lemma.

Lemma 3

Let $0 < \delta \le \delta_0$. For any $\varepsilon > 0$ and any integer $p \ge 1$, when t goes to infinity:

$$\sup_{x\ne y}\frac{\|L(t,x)-L(t,y)\|_{2p}^{2p}}{|x-y|^{2p\delta}} = o(t^{2p(1-\tau(1+\delta))+\varepsilon})$$

Lemma 4

Let $0 < \gamma < \delta \le \delta_0$. For any $\varepsilon > 0$ and any integer $p \ge 1$, when t goes to infinity:

$$\sup_{x\in\mathbb{R}}\left\|\int_0^1 l(y)\frac{L(t,x+y)-L(t,x-y)}{y^{1+\gamma}}dy\right\|_{2p}^{2p} = o(t^{2p(1-\tau(1+\delta))+\varepsilon})$$

Proof

We have:

$$\sup_{x\in\mathbb{R}}\left\|\int_0^1 l(y)\frac{L(t,x+y)-L(t,x-y)}{y^{1+\gamma}}dy\right\|_{2p}^{2p}$$
$$\le \sup_{x\in\mathbb{R}}\sup_{0<y\le 1}\frac{\|L(t,x+y)-L(t,x-y)\|_{2p}^{2p}}{y^{2p\delta}}\left|\int_0^1\frac{l(y)}{y^{1+\gamma-\delta}}dy\right|^{2p}$$

By virtue of Lemma 3 and the fact that l is bounded on [0, 1], we deduce the lemma.

Lemma 5

Let $0 < \delta \le \delta_0$. For any $\varepsilon > 0$ and any integer $p \ge 1$, when t goes to infinity:

$$\sup_{|x|\le t^\alpha}\left\|\int_1^\infty l(y)\frac{L(t,x+y)-L(t,y)}{y^{1+\gamma}}dy\right\|_{2p}^{2p} = o(t^{2p(1-\tau(1+\delta))+2pa\delta+\varepsilon})$$

for some a > 0.

Proof

We have for any $0 < \delta \le \delta_0$:

$$\sup_{|x|\le t^a}\left\|\int_1^{\infty} l(y)\frac{L(t,x+y)-L(t,y)}{y^{1+\gamma}}dy\right\|_{2p}^{2p}$$
$$\le \sup_{|x|\le t^a}\sup_{y\in\mathbb{R}}\|L(t,x+y)-L(t,y)\|_{2p}^{2p}\left|\int_1^{\infty}\frac{l(y)}{y^{1+\gamma}}dy\right|^{2p}$$
$$\le \sup_{|x|\le t^a}|x|^{2p\delta}\sup_{y\in\mathbb{R}}\frac{\|L(t,x+y)-L(t,y)\|_{2p}^{2p}}{|x|^{2p\delta}}\left|\int_1^{\infty}\frac{l(y)}{y^{1+\gamma}}dy\right|^{2p}$$

Using Potter's Theorem for $x = 1$, $y \ge 1$ and $0 < \xi < \gamma$, we obtain:

$$\int_1^{+\infty}\frac{l(y)}{y^{1+\gamma}}dy < \infty \tag{9}$$

Finally, by virtue of (9) and Lemma 3, we deduce the desired estimate.

Lemma 6

Under same conditions as in Theorem 4. For any sufficiently small $\varepsilon > 0$ and any integer $p \ge 1$, when t goes to infinity, we have:

$$\left\|\int_0^t K^{1,\gamma}f(Y_s^{\tau})ds\right\|_{2p} = \frac{I(f)}{\Gamma(1-\gamma)}\left\|K^{1,\gamma}L(t,.)(0)\right\|_{2p} + o(t^{1-\tau(1+\gamma)-\varepsilon})$$

where, $0 < \gamma < \delta$.

Proof

The proof is similar to that given by Ouahra and Ouali (2009) in the case of fractional derivatives. Indeed, by the occupation density formula and (6), we have:

$$I(t) := \left\|\int_0^t K^{1,\gamma}f(Y_s^{\tau})ds - \frac{I(f)}{\Gamma(1-\gamma)}K^{1,\gamma}L(t,.)(0)\right\|_{2p}^{2p}$$
$$= C\left\|\int_{\mathbb{R}}(K^{1,\gamma}L(t,.)(x) - K^{1,\gamma}L(t,.)(0))f(x)dx\right\|_{2p}^{2p}$$
$$\le C(I_1(t) + I_2(t))$$

Where:

$$I_1(t) := \left\|\int_{|x|>t^a}(K^{1,\gamma}L(t,.)(x) - K^{1,\gamma}L(t,.)(0))f(x)dx\right\|_{2p}^{2p}$$

and:

$$I_2(t) := \left\|\int_{|x|\le t^a}(K^{1,\gamma}L(t,.)(x) - K^{1,\gamma}L(t,.)(0))f(x)dx\right\|_{2p}^{2p}$$

for some $0 < a \le \tau$.

Let us deal with the first term $I_1(t)$. Lemma 2 and (8) imply that:

$$I_1(t) \le \sup_{|x|>t^a}\left\|K^{1,\gamma}L(t,.)(x) - K^{1,\gamma}L(t,.)(0)\right\|_{2p}^{2p}\left|\int_{|x|>t^a}|x|^{-k}|x|^k|f(x)|dx\right|^{2p}$$
$$\le t^{-2pak}\sup_{|x|>t^a}\left\|K^{1,\gamma}L(t,.)(x) - K^{1,\gamma}L(t,.)(0)\right\|_{2p}^{2p}\left|\int_{|x|>t^a}|x|^k|f(x)|dx\right|^{2p}$$
$$= o(t^{2p(1-\tau(1+\gamma))-2pak+\varepsilon})$$

Now, we deal with $I_2(t)$. By the definition of $K^{1,\gamma}$ and the fact that f is integrable, we have:

$$I_2(t) \le \sup_{|x|\le t^a}\left\|\int_0^{\infty} l(y)\frac{L(t,x+y)-L(t,x-y)-L(t,y)+L(t,-y)}{y^{1+\gamma}}dy\right\|_{2p}^{2p}\left|\int_{|x|\le t^a}|f(x)|dx\right|^{2p}$$
$$\le C\sup_{|x|\le t^a}\left\|\int_0^{1} l(y)\frac{L(t,x+y)-L(t,x-y)-L(t,y)+L(t,-y)}{y^{1+\gamma}}dy\right\|_{2p}^{2p}$$
$$+C\sup_{|x|\le t^a}\left\|\int_1^{\infty} l(y)\frac{[L(t,x+y)-L(t,y)]-[L(t,x-y)]-L(t,-y)}{y^{1+\gamma}}dy\right\|_{2p}^{2p}$$

which, in view of Lemmas 4 and 5, implies:

$$I_2(t) = o(t^{2p(1-\tau(1+\delta))+\varepsilon}) + o(t^{2p(1-\tau(1+\delta))+2pa\delta+\varepsilon})$$
$$= o(t^{2p(1-\tau(1+\delta))+2pa\delta+\varepsilon})$$

Then:

$$I(t) = o(t^{2p(1-\tau(1+\gamma))-2pka+\varepsilon}) + o(t^{2p(1-\tau(1+\delta))+2pa\delta+\varepsilon})$$

choosing:

$$a = \frac{\tau(\delta-\gamma)}{\delta+k}$$

It is clear that $0 < a \le \tau$. We finally get:

$$I(t) = o(t^{2pb+\varepsilon})$$

With:

$$b = \frac{\delta(1-\tau(1+\gamma)) + k(1-\tau(1+\delta))}{k+\delta}$$

Clearly b < 1-τ (1 + γ), because γ < δ. Then for all sufficiently small ε > 0, when t goes to infinity:

$$I(t) = o(t^{2p(1-\tau(1+\gamma))-\varepsilon})$$

which gives the desired estimate.

We will also need the following estimate between fractional derivative D^γ and generalized fractional derivative $K^{l,\gamma}$.

Lemma 7

Let f be a Borel function on R satisfying (8) for some k > 0. Then, for any sufficiently small ε > 0 and any integer p ≥ 1, when t goes to infinity, we have:

$$\frac{I(f)}{\Gamma(1-\gamma)}\left[\left\|K^{l,\gamma}L(t,.)(0)\right\|_{2p} - \left\|D^\gamma L(t,.)(0)\right\|_{2p}\right] = o(t^{1-\tau(1+\gamma)-\varepsilon})$$

Proof

We have by (8):

$$J(t) := \frac{I(f)}{\Gamma(1-\gamma)}\left\|K^{l,\gamma}L(t,.)(0)\right\|_{2p} \le C(J_1(t) + J_2(t))$$

Where:

$$J_1(t) := \sup_{|x|>t^a}\left\|K^{l,\gamma}L(t,.)(0)\right\|_{2p}\int_{|x|>t^a}|x|^{-k}|x|^k|f(x)|dx$$

And:

$$J_2(t) := \sup_{|x|\le t^a}\left\|K^{l,\gamma}L(t,.)(0)\right\|_{2p}$$

The same arguments used in the proof of Lemma 6 implies that:

$$J_1(t) = o(t^{1-\tau(1+\gamma)-ka+\varepsilon})$$

For $J_2(t)$, we have by Lemma 2:

$$J_2(t) = o(t^{1-\tau(1+\delta)+\varepsilon})$$

Therefor:

$$J_2(t) = o(t^{1-\tau(1+\delta)+a\delta+\varepsilon})$$

for any a>0.

Consequently:

$$J(t) = o(t^{1-\tau(1+\gamma)-\varepsilon})$$

In particular, if we take $l \equiv 1$, we get:

$$\frac{I(f)}{\Gamma(1-\gamma)}\left\|D^\gamma L(t,.)(0)\right\|_{2p} = o(t^{1-\tau(1+\gamma)-\varepsilon})$$

The proof of Lemma 7 is done.

Now, we return to the proof of Theorem 4.

Proof of Theorem 4

This theorem is an immediate consequence of Lemma 6 and Lemma 7.

Remark 8

- In case f is the fractional derivative of some function g, the analogous results of Theorem 4 appeared in Ouahra and Ouali (2009). On the other hand, the a.s. estimate of Theorem 4 is given in Csaki *et al*. (2000) for special Bm case
- We should point out that in this paper we only study the L^p-estimate of our limit theorems. This is enough for the purpose of this study. We will study the a.s. estimates in future work and apply this idea to study the law of the iterated logarithm of stochastic process of the form $\int_0^t K^{l,\gamma}(Y_s^\tau)ds$

2. ACKNOWLEDGMENT

This study was completed while the first authors was visiting the Department of Statistics and Probability at Michigan State University (DSPMSU). This author would like to express his sincere thanks to the staff of DSPMSU for generous supported and hospitality, especially Prof. Yimin Xiao. He was also supported by Fulbright Visiting Scholar grant 2012-2013.

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