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A Kind of Intersection Graphs on Ideals of a Ring

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Abstract: Problem statement: Let R be a ring. The graph G(R) is the graph whose vertices are nontrivial ideal of R and in which two vertices u, v are joined by an edge, if and only if u ∩ v #{0}. **Approach:** In this study we study some properties of G(R). **Results:** We obtain conditions of R such that G(R) is a path and determine the graph G(R) in which it is a tree. **Conclusion:** We conclude that ideals of R have degree one.

Key words: Graphs related, intersection graph, integers modulo, algebraic structure, distinct vertices, intersection graphs, obtain conditions

INTRODUCTION

Let $A = \{S_i: i\in I\}$ be an arbitrary family of sets. The intersection graph G (A) is the graph whose vertices are S_i, $i\in I$ and in which the vertices S_i, S_j are adjacent if and only if $S_i\neq S_j$ and $S_i\cap S_j\neq\phi$. It is more interesting to study the intersection graphs G(A) when the elements of A have an algebraic structure. These terdisciplinary studies allow us to obtain characterization and representation of especial classes of algebraic structure in terms of graphs and vice versa.

Many authors studied such graphs related to the group structure, for example Shen (2010) and Zelinka (1975). Various construction of graphs relative to the ring structure are found in Simis *et al*. (1994). Chakrabarty *et al.* (2009) studied the intersection graphs of ideal of rings. They determined the values of n for which $G(Z_n)$ for ring Z_n of integers modulo n for $n\in N$ is connected, complete, bipartite, Eulerian and Hamiltonian.

For a given graph H, the degree of a vertex V in H denoted by deg (v), is the number of edges incident to v. A path P is a sequence of distinct vertices v_1, v_2, $\dots v_{m+1}$ in which very two consecutive vertices are adjacent. The number m is called the length of P. For two vertices u and V in a graph H the distance between u and v, denoted by d (u, v), is the length of the shortest path between u and v, if such a path exist; otherwise we define d (u, v) = ∞.

A graph H is connected if there is a path between each pair of the vertices of H The diameter diam (H) of a connected graph H is the maximum of d (u, v) overall vertices u, v The path v_1, v_2, $\dots v_{m+1}$ is called a cycle if $v_1= v_{m+1}$ A graph H is a tree if it be connected and have no cycle. A graph H with vertex set Eq. 1:

$$\{u, v_1, \dots, v_m\} \tag{1}$$

And edge set $E=\{\{u,v_i\}:1\leq i\leq m$ is called a star graph.

In this study, we determine conditions on ring R such that G(R) is a tree or a path and get ideals of R who's these degree is one.

MATERIALS AND METHODS

Consider the ring Z_n of integers modulo n for $n\in N$. We know that Z_n is a principle ideal ring and each of these ideals is generated by $m \in Z_n$where m is a factor of n. Let M denoted the set of all positive integers greater than one and which are not primes. In the following we have:

Theorem 1: [1] Let $n\in$ M. The graph G (Z_n) is complete if and only if $n = p^m$ where P is a prime number and $m \in N$, m>1.

We investigate the diameter G (Z_n).

Obviously we have the following theorem:

- Theorem 2.2. For any $n\in M$, diam $(G(Z_n))\leq 2$
- So clearly by Theorem 2.1 we have the following corollary
- Corollary 2.3. Let $n\in M$, then diam $(G(Z_n))$ = 2if and only if $n \neq p^m$

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RESULTS

We state the essential results of this study.

Proposition 3.1: Let I be a minimal ideal of a ring R. Let graph G(R) have no cycle of length 3. Then we have deg(I) =1,in G(R) .

Proof: Let I be a minimal ideal of a ring R and deg(I) ≥2. Let J and K are two distinct vertices of G(R) such that both J and K are adjacent with vertex I Then $I\cap J\neq (0)$ and $I\cap K \neq (0)$. Thus $I\cap J=I$, because $I\bigcap J\subseteq I$ and I is minimal. Hence $I\subseteq J$. Similarly we get $I\subseteq k$. Thus $I\subseteq J\cap k$ and this implies that J and K are adjacent. Thus the sequence I, J, K of vertices of G(R) is a cycle of length 3, a contradiction.

Theorem 3.2: For ring $G(R) \approx P_2$, if and only if R have only two ideal, namely minimal ideal and maximal ideal.

Proof: Let $G(R) \approx P_2$ and J,K are two vertices of G(R) Then $I\cap J = I$ or $I\cap J = J$. If $I\cap J = I$, then $I\subseteq J$. This implies that I is minimal ideal and J is maximal ideal. If $I\cap J = J$, similarly we conclude that J is minimal ideal and I is maximal ideal. Conversely, for nontrivial graph G(R) is obvious.

Example: For ring $R= Z_{p3}$, there is only two nontrivial ideal I = (a), J= (a^2) and we have $G(R) \approx P_2$.

Lemma 3.3: Let I be a vertex of G(R) such that deg (I)=1. Then I is a minimal ideal or maximal ideal.

Proof: Let for vertex I, deg (I)=1 and J be only vertex of G(R) such that J is adjacent to I. Then by definition, $I\bigcap J\neq(0)$. Since deg (I)=1, $I\cap J=I$ or , $I\cap J=J$. Thus $I\subseteq J$ or $J\subseteq I$. If $I\subseteq J$, then there is not nontrivial ideal L such that $L\subset I$, because, deg (I)=1. Hence I is minimal ideal. If $J\subset I$, then there is not ideal L such that $I\subset L \subset R$, because deg (I) = 1. Thus I is maximal.

Lemma 3.4: Let G(R) is a path as sequence $I_1, I_2,..I_t$. If I_1be a maximal ideal of R, then $G(R) \approx P_2$.

Proof: Let I_1be maximal ideal. We know that $I_1\cap I_2 = I_1$or vertices $I_1\cap I_2$ and I_1 are adjacent. If $I_1\cap I_2 = I_1$, then $I_2=I_1$, because $I_1\subseteq I_2$ and I_1 is maximal. This is a contradiction with $I_2 \neq I_1$. Thus $I_1\cap I_2 = I_2$ and so $I_2 \subseteq I_1$. Let n ≥ 3. Therefore $(0)\neq I_2 \cap I_3 = I_2$ or vertices $I_2 \cap I_3$ and 1_2 are adjacent. If $(0)\neq I_2 \cap I_3=I_2$, then $I_2 \subseteq I_3$ and so $I_2 \subseteq I_3 \cap I_1$. Thus vertices I_1 and I_3 are adjacent, a contradiction. Otherwise we have $I_2 \cap I_3= I_1$or $I_2 \cap I_3= I_3$. If $I_2 \cap I_3= I_3$, then $I_1$and $I_3$are adjacent, a contradiction. But if $I_2 \cap I_3= I_3$, then $I_3 \subseteq I_2$. Now in view of $I_2\subseteq I_1$we get that vertices I_1 and I_3 are adjacent, a contradiction.

Theorem 3.5: Let G(R) is a path, then $G(R)\cong P_2$ or $G(R) \approx P_3$. If $G(R) \approx P_3$, then R have only three ideals I_1,I_2,I_3 such that $I_2=I_1\oplus I_3$.

Proof: Let G(R) be a path as sequence $I_1,I_2,\dots I_n$. By Lemma 3.3, the ideal $I_1$is a minimal ideal or maximal ideal. Let I_1be a minimal ideal. We have $I_1\cap I_2\neq(0)$, because vertices $I_1$and $I_2$are adjacent. Thus $I_2 \cap I_3= I_1$, because $I_1 \cap I_2 \subseteq I_1$and I_1 is a minimal ideal and hence $I_1\subseteq I_2$. Let n>2, then $I_2 \cap I_3\neq I_1$, because $I_2\cap I_3= I_2$ implies $I_2\subseteq I_3$ and in view of $I_1\subseteq I_2$, we conclude that $I_1$and I_3 are adjacent, that is a contradiction. Hence $I_2 \cap I_3= I_1$or $I_2\bigcap I_3 = I_3$. If $I_2 \cap I_3= I_1$, then I_1 and I_3 are adjacent, a contradiction. Thus $I_2 \cap I_3= I_1$and so $I_3\subseteq I_2$. Thus, this is true for n=3. Let n>3. If $I_4\cap I_3= I_3$, in view of $I_3\subseteq I_2$ we conclude that I_3 and I_4 are adjacent, a contradiction. Thus vertices $I_3\cap I_4$ and I_3 are adjacent. So, $I_4\cap I_3=I_2$or $I_4\cap I_3=I_4$. If $I_4\cap I_3=I_4$, then $I_4\subseteq I_3$ and so $I_4 \subseteq I_3 \subseteq I_2$. This implies that I_2 and I_4 are adjacent, a contradiction. If $I_4\bigcap I_3 = I_2$, then $I_1 \subseteq I_2 \subseteq I_3$. Thus I_1 and I_3 are adjacent, a contradiction. In final, by conditions $I_1\subseteq I_2$, $I_3\subseteq I_2$ and this fact that I_1 and I_3 are not adjacent we conclude $I_2=I_1 \oplus I_3$.

If I_1 be maximal ideal we conclude the theorem by Lemma 3.4.

By a similar argument as in the proof of Theorem 3.5 we see that in a tree, length of a path with end points of degree 1 is at most 2. Thus we obtain the following Corollary:

Corollary 3.6: For a ring R, the graph G(R) is a tree if and only if is a star graph.

DISCUSSION

We studied a especial class of a algebraic structure (the ring structure) in terms of graphs and vice versa. Using the properties of ideals in a ring, we discussed some of the properties of the graph G(R).

CONCULSION

Using notions graph theory and group theory we characterized the graph G(R), when it is a path or tree, determined ideals I, when deg (I)=1 in the graph G(R). Also we obtained a bound for diam ($G(Z_n)$).

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