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A Note on Improved Young Type Inequalities with Kantorovich Constant

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Abstract: In this article, we first present some improved Young type inequalities for scalars, then according to these inequalities we give the Hilbert-Schmidt norm and the trace norm versions.

Keywords: Young Type Inequalities, Hilbert-Schmidt Norm, Positive Semidefinite Matrices, Singular Values, Refinement

Introduction

The scalar Young inequality says that if a and b be nonnegative real numbers and $0 \le v \le 1$, then:

$$a^v b^{1-v} \le va + (1-v)b \tag{1.1}$$

With equality if and only if $a = b$: Inequality (1.1) is called the ν-weighted arithmetic-geometric mean inequality. If $v = \frac{1}{2}$ then:

$$\sqrt{ab} \le \frac{a+b}{2} \tag{1.2}$$

Which is called arithmetic-geometric mean.

Zuo *et al*. (2011) refined Young inequality (1.1) as follows:

$$va + (1-v)b \ge K(h,2)^{\Upsilon} a^v b^{1-v} \tag{1.3}$$

For all a, $b \ge 0$ and $\nu \in [0, 1]$, where $r = min\ \{v, 1-v\}$, $h = \frac{a}{b}$ and $K(h, 2) = \frac{(h=2)^2}{4h}$, so that $K(h, 2)$ is called Kantorovich constant.

Here, we need to recall that Kantorovich constant satisfies the following properties:

(i) $K(1, 2) = 1$,

(ii) $K(t, 2) = K\left(\frac{1}{t}2\right)$

(iii) $K(t, 2)$ is monotone increasing on $[1, \infty)$ and is monotone decreasing on $(0, 1]$

Wu and Zhao (2013) improved inequality (1.3) in the following form:

$$v_a + (1-v)b \ge \Upsilon(\sqrt{a}-\sqrt{b})^2 + K(\sqrt{h},2)^{\Upsilon'} a^v b^{1-v} \tag{1.4}$$

For all a, $b \ge 0$ and $\nu \in [0, 1]$; where $h = \frac{a}{b}$, $r = min\ \{v, 1-v\}$ and $r' = min\{2r, 1-2r\}$.

By (Nasiri *et al*., 2016) we give some improved inequalities of (1.1) as follows:

$$v^2a^2 + (1-v)^2b^2 \ge (va)^{2v}b^{2-2v} + v^2(a-b)^2 + r_0 b(\sqrt{va}-\sqrt{b})^2,\ 0 \le v \le \frac{1}{2} \tag{1.5}$$

where, $h = \frac{va}{b}$ and $r = min\{2\nu, 1-2\nu\}$.

And:

$$v^2a^2 + (1-v)^2b^2 \ge a^{2v}[(1-v)b]^{2-2v} + (1-v)^2(a-b)^2 + r_0 a(\sqrt{a}-\sqrt{(1-v)b})^2, \frac{1}{2} \le v \le 1 \tag{1.6}$$

where, $h = \frac{a}{(1-v)b}$ and $r_0 = min\{2\nu - 1, 2(1-\nu)\}$.

Suppose M_n be the space of $n \times n$ complex matrices. Suppose $\|.\|$ denote any unitarily invariant norm on M_n. So, $\|UAV\| = \|A\|$ for all $A \in M_n$ and for all unitary matrices U, $V \in M_n$. For $A = [a_{ij}] \in M_n$, the Hilbert-Schmidt norm (or Frobenious) and the trace norm of A are defined by:

$$\| A \|_2 = \sqrt{\sum_{j=1}^{n} S_j^2(A)} \quad \| A \|_1 = \sum_{j=1}^{n} S_j(A)$$

Respectively, where $s_1(A) \geq s_2(A) \geq \ldots\ldots \geq s_n(A)$ are the singular values of A, that is, the eigenvalues of the positive matrix $|A| = (A^*A)^{\frac{1}{2}}$ arranged in decreasing order and repeated according to multiplicity. It is known that the Hilbert-Schmidt norm is unitarily invariant.

For more information about Young type inequality and its matrix version the reader is referred to (Bhatia, 1996; Hu, 2012).

Inequalities for Scalars

In this section, we will derive some Young type inequalities for scalars.

Theorem 2.1. Let $a, b \geq 0$ and $0 \leq v \leq 1$.

(i) If $0 \leq v \leq \frac{1}{2}$, then:

$$\begin{aligned} &v^2a^2 + (1-v)^2b^2 \geq v^2(a-b)^2 \\ &+rb(\sqrt{b} - \sqrt{va})^2 + K(\sqrt{hv}, 2)^{r'}[(va)vb^{1-v}]^2 \end{aligned} \tag{2.1}$$

where, $h = \frac{a}{b}$, $r = min\{2v, 1\text{-}2v\}$ and $r' = min\{2r, 1-2r\}$.

(ii) If $\frac{1}{2} \leq v \leq 1$, then

$$\begin{aligned} &v^2a^2 + (1-v)^2b^2 \geq (1-v)^2(a-b)^2 + ra(\sqrt{a} - \sqrt{(1-v)b})^2 \\ &+K(\sqrt{\frac{h}{1-v}}, 2)^{r'}[a^v((1-v)b)^{1-v}]^2 a^{2v} \end{aligned} \tag{2.2}$$

where, $h = \frac{a}{b}$, $r = min\{2v - 1, 2(1-v)\}$ and $r' = min\{2r, 1 - 2r\}$.

Proof: Let $0 \leq v \leq \frac{1}{2}$. Then we have:

$$\begin{aligned} &v^2a^2 + (1-v)^2b^2 - v^2(a-b)^2 \\ &= b[(1-2v)b + 2v(va)] \\ &\geq b[r(\sqrt{b} - \sqrt{va})^2 + K(\sqrt{vh}, 2)r' b^{1-2v}(va)^{2v}] \\ &= rb(\sqrt{b - \sqrt{va}})^2 + K(\sqrt{vh}, 2)r' b^{2-2v}(va)^{2v} \end{aligned} \tag{1.4}$$

That is:

$$\begin{aligned} v^2a^2 + (1-v)^2b^2 &\geq v^2(a-b)^2 + rb(\sqrt{b} - \sqrt{va})^2 \\ &+ K(\sqrt{hv}, 2)r'[(va)^v b^{1-v}]^2 \end{aligned}$$

Thus, (2.1) holds.

For $\frac{1}{2} \leq v \leq 1$, compute:

$$\begin{aligned} &v^2a^2 + (1-v)^2b^2 - (1-v)^2(a-b)^2 \\ &= a[(2v-1)a + 2(1-v)((1-v)b)] \\ &\geq a\begin{bmatrix} r\left(\sqrt{a} - \sqrt{(1-v)b}\right)^2 \\ +K\left(\sqrt{\frac{h}{1-v}}, 2\right) r' a^{2v-1}(b(1-v)^{2(1-v)} \end{bmatrix} \\ &= rb(\sqrt{a} - \sqrt{(1-v)b})^2 + K(\sqrt{\frac{h}{1-v}}, 2)r'[a^v(b(1-v))^{1-v}]^2 \end{aligned} \tag{1.4}$$

Hence:

$$\begin{aligned} &v^2a^2 + (1-v)^2b^2 \geq (1-v)^2(a-b)^2 + ra(\sqrt{a} - \sqrt{(1-v)b})^2 \\ &K\left(\sqrt{\frac{h}{1-v}}, 2\right)r'[a^v(b(1-v)^{1-v}]^2 \end{aligned}$$

This estimate completes the proof of (2.2); Theorem 2.1 is thereby proved.

Remark 1: Clearly, inequalities (2.1) and (2.2) are improvement of inequalities (1.5) and (1.6).

Inequalities for Matrices

According to the results obtaining from section 2, we present the trace and the Hilbert-Schmidt norm versions.

Let $A, B, X \in M_n$ so that A and B are positive semidefinite. We recall since $A, B > 0$, thus $A = UDU^*$ and $B = V\,EV^*$ where $D = diag(\lambda_1, \ldots, \lambda_n)$ and $E = diag(\mu_1, \ldots, \mu_n)$, λi, $v_i \geq 0$ for $1 \leq i \leq n$. Moreover, we mention that λ_i and μ_i are the singular values of A and B (respectively). It is showed (Nasiri *et al.*, 2016) that:

$$\begin{aligned} &\left\| \upsilon AX + (1 - v XB \right\|_2^2 + v^{2v} \left\| A^v XB^{1-v} \right\|_2^2 + v^2 \Big\| \\ &AX - XB \Big\|_2^2 + r_0 [v \left\| A^{\frac{1}{2}} XB^{\frac{1}{2}} \right\|_2^2 + \\ &\left\| XB \right\|_2^2 - 2\sqrt{v} \left\| A^{\frac{1}{4}} XB^{\frac{3}{4}} \right\|_2^2] + 2v(1-v) \left\| A^{\frac{1}{2}} XB^{\frac{1}{2}} \right\|_2^2 \end{aligned} \tag{3.1}$$

where, $K = \min\left\{\left(k\frac{v\lambda_i}{\mu_j}, 2\right) i, j = 1, \ldots, n\right\}$ and $r = \min\{2v, 2v-1\}$:

$$\begin{aligned} &\left\| \upsilon AX + (1 - v XB \right\|_2^2 \geq (1-v)^{2-2v} \left\| A^v XB^{1-v} \right. \\ &\quad \Big\|_2^2 + (1-v)^2 \left\| AX - XB \right\|_2^2 + \\ &r_0 [(1-v) \left\| A^{\frac{1}{4}} XB^{\frac{1}{2}} \right\|_2^2 + \left\| AX \right\|_2^2 - 2\sqrt{1-v} \left\| A^{\frac{3}{4}} XB^{\frac{1}{4}} \right. \\ &\quad \Big\|_2^2] + 2v(1-v) \left\| A^{\frac{1}{2}} XB^{\frac{1}{2}} \right\|_2^2 \end{aligned} \tag{3.2}$$

where, $K = \min\{(k\frac{\lambda_i}{(1-v)\mu_j},2)i,j=1,....,n\}$ and $r = \min\{2v-1,\ 2-2v\}$.

Our first main result is the following.

Theorem 3.1. Let A, B, $X \in M_n$ so that A and B are positive semidefinite.

(i) If $0 \le v \le \frac{1}{2}$, then:

$$\left\|\upsilon AX+(1-v)XB\right\|_2^2 \ge v^2\left\|AX-XB\right\|_2^2 + r[\left\|XB\right\|_2^2 + v\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2 - 2\sqrt{v}\left\|A^{\frac{1}{4}}XB^{\frac{3}{4}}\right\|_2^2] + Kr'v^{2v}\left\|A^vXB^{1-v}\right\|_2^2 + 2v(1-v)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2 \quad (3.3)$$

where, $K = \min\left\{K\left(\sqrt{\frac{v\lambda_i}{\mu_j}},2\right), 1 \le i,j \le n\right\}, r = \min\{2v, 1-2v\}$ and $r' = \min\{2r, 1-2r\}$.

(ii) If $\frac{1}{2} \le v \le 1$, then:

$$\left\|\upsilon AX+(1-v)XB\right\|_2^2 \ge (1-v)^2\left\|AX-XB\right\|_2^2 + r[\left\|AX\right\|_2^2 + (1-v)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2 - 2\sqrt{1-v}\left\|A^{\frac{3}{4}}XB^{\frac{1}{4}}\right\|_2^2)] + K^{r'}(1-v)^{2(1-v)}\left\|A^vXB^{1-v}\right\|_2^2 + 2v(1-v)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2 \quad (3.4)$$

where, $K = \min\left\{K\left(\sqrt{\frac{\lambda_i}{(1-v)\mu_j}},2\right), 1 \le i,j \le n\right\}$, $r = \min\{2v-1, 2(1-v)\}$ and $r' = \min\{2r, 1-2r\}$.

Proof: To prove assertions of Theorem 3.1, we need to obtain assertions $\upsilon AX+(1-v)XB, AX-XB, A^{\frac{1}{2}}XB^{\frac{1}{2}}$ and A^vXB^{1-v}. It is well known, that every positive semidefinite matrix is unitarily diagonalizable. Therefore, it concludes that there are unitary matrices $U, V \in M_n$ so that $A = UDU^*$ and $B = VEV^*$, where $D = diag(\lambda_1,...,\lambda_n)$ and $E = diag(\mu_1,...,\lambda_n)$ with $\lambda_i, \mu_i \ge 0$ for $1 \le i \le n$.

Let $Y = U^*XV = [y_{ij}]$, then we have:

$$\begin{aligned} A^vXB^{1-v} &= (UDU^*)^v X(VEV^*)^{1-v} \\ &= UD^v(uU^*XV)E^{1-v}V^* \\ &= U(D^vYE^{1-v})V^* \end{aligned}$$

Thus, using unitarily invariant property of $\|.\|_2$, it follows that:

$$\begin{aligned} \left\|A^vXB^{1-v}\right\|_2^2 &= \left\|U(D^vYE^{1-v})V^*\right\|_2^2 \\ &= \left\|D^vYE^{1-v})V^*\right\|_2^2 \\ &= \sum_{i,j=1}^n(\lambda_i^v v_j^{1-v})^2 |y_{ij}|^2 \end{aligned}$$

Analogously, we can prove that:

$$\upsilon AX+(1-v)XB = U[(v\lambda_i+(1-v)\mu_j)y_{ij}]V^*,$$
$$AX-XB = U[(\lambda_i-\mu_j)y_{ij}]V^*$$

And:

$$A^{\frac{1}{2}}XB^{\frac{1}{2}} = U[(\lambda_i^{\frac{1}{2}}\mu_j^{\frac{1}{2}})y_{ij}]V^*$$

We firstly suppose $0 \le v \le \frac{1}{2}$, then in view of the unitary invariance of the Hilbert-Schmidt norm, we have:

$$\begin{aligned} &\left\|\upsilon AX+(1-v)XB\right\|_2^2 \\ &= \sum_{i,j=1}^n(v\lambda_i+(1-v)\mu_j)^2|y_{ij}|^2 \\ &\ge v^2\sum_{i,j=1}^n(\lambda_i-\mu_j)^2|y_{ij}|^2 \\ &+r\mu_j(\sqrt{\mu_j}-\sqrt{v\lambda_i})^2|y_{ij}|^2 \\ &+Kr'v^{2v}\sum_{i,j=1}^n(\lambda_i^v\mu_j^{1-v})^2|y_{ij}|^2+2v(1-v) \\ &\sum_{i,j=1}^n(\lambda_i^{\frac{1}{2}}\mu_j^{\frac{1}{2}})^2|y_{ij}|^2 \\ &= v^2\left\|AX-XB\right\|_2^2+r[\left\|XB\right\|_2^2+v\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2-2\sqrt{v}\left\|A^{\frac{1}{4}}XB^{\frac{3}{4}}\right\|_2^2] \\ &+Kr'v^{2v}\left\|A^VXB^{1-V}\right\|_2^2+2v(1-v)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2 \end{aligned} \quad (2.1)$$

i.e.,

$$\left\|\upsilon AX+(1-v)XB\right\|_2^2 \ge v^2\left\|AX-XB\right\|_2^2+r[\left\|XB\right\|_2^2+v\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2-2\sqrt{v}\left\|A^{\frac{1}{4}}XB^{\frac{3}{4}}\right\|_2^2] +Kr'v^{2v}\left\|A^VXB^{1-V}\right\|_2^2+2v(1-v)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2$$

This show (3.1) holds.

Using of unitarily invariant property $\|.\|_2$ and by (2.2); we can obtain (3.4).

This completes the proof.

Remark 2: Obviously, (3.3) and (3.4) are improvement of inequalities (3.1) and (3.2).

At the end, we recall the following Lemmas that are necessary to obtain the other inequalities by (2.1) and (2.2).

Lemma 3.1. Bhatia (1996) (Cauchy-Schwarz inequality)*:* Suppose a_i, $b_i \geq 0$, $(1 \leq i \leq n)$. Then:

$$\sum_{i=1}^{n} a_i b_i \leq (\sum_{i=1}^{n} a_i^2)^{\frac{1}{2}} (\sum_{i=1}^{n} b_i^2)^{\frac{1}{2}} \tag{3.5}$$

Lemma 3.2. Bhatia (1996) Let A, $B \in M_n$, then:

$$\sum_{i=1}^{n} s_j(AB) \leq (\sum_{i=1}^{n} s_j(A) s_j(B) \tag{3.6}$$

Theorem 3.2. Let A, $B \in M_n$ so that A and B are positive semidefinite and $0 \leq v \leq 1$.

(i) If $0 \leq \upsilon \leq \frac{1}{2}$ then:

$$\begin{aligned} & v^2 \|A\|_2^2 + (1-v)^2 \|B\|_2^2 \geq v^2 [\|A\|_2^2 + \|B\|_2^2 - 2\|AB\|_2^2] \\ & + r[\|B\|_2^2 + v\|AB\|_1 - 2\sqrt{v}\sqrt{\|A\|_1}\sqrt{\|B^3\|_1}] \\ & + Kr' v^{2v} \|A^v B^{1-v}\|_2^2 \end{aligned} \tag{3.7}$$

where, $K = \min\left\{ k\left(\sqrt{\frac{v\lambda_i}{\mu_j}}, 2 \right), 1 \leq i, j \leq n \right\}$, $r = \min\{2v, 1-2v\}$ and $r' = \min\{2r, 1-2r\}$.

(ii) If $\frac{1}{2} \leq v \leq 1$, then we have:

$$\begin{aligned} & v^2 \|A\|_2^2 + (1-v)^2 \|B\|_2^2 \geq (1-v)^2 [\|A\|_2^2 + \|B\|_2^2 - 2\|AB\|_1] \\ & + r[\|A\|_2^2 + (1-v)\|AB\|_1 - 2\sqrt{1-v}\sqrt{\|B\|_1}\sqrt{\|A^3\|_1}] \\ & + Kr'(1-v)^{2(1-v)} \|A^{1-v} B^v\|_2^2 \end{aligned} \tag{3.8}$$

where,

$K = \min\left\{ k\left(\sqrt{\frac{\lambda_i}{(1-v)\mu_j}}, 2 \right), 1 \leq i, j \leq n \right\}$, $r = \min\{2v-1, 2v(1-v)\}$ and $r' = \min\{2r, 1-2r\}$.

Proof: Here, to establish Theorem 3.2*,* we will first prove the assertion of Theorem 3.1 together with estimate 3.5.

For $0 \leq v \leq \frac{1}{2}$, based on inequalities (2.1), (3.5) and (3.6), it follows that:

$$\begin{aligned} & tr(v^2 A^2 + (1-v)^2 B^2) \\ & = \upsilon^2 trA^2 + (1-v)^2 trB^2 \\ & = \sum_{j=1}^{n} (v^2 s_j^2(A) + (1-v)^2 s_j^2(B)) \\ & \geq v^2 [\sum_{j=1}^{n} s_j^2(A) + \sum_{j=1}^{n} s_j^2(B) - 2\sum_{j=1}^{n} s_j(A) s_j(B)] \\ & + r\sum_{j=1}^{n} s_j(B)[\sqrt{s_j(B)} - \sqrt{vs_j(A)}]^2 \\ & + K(\sqrt{\frac{vs_j(A)}{s_j(B)}}, 2) r' v^{2v} \sum_{j=1}^{n} [s_j(A^v) s_j(B^{1-v})]^2 \\ & \geq v^2 [\sum_{j=1}^{n} s_j^2(A) + \sum_{j=1}^{n} s_j^2(B) - 2\sum_{j=1}^{n} s_j(AB)] \\ & + r[\sum_{j=1}^{n} s_j^2(B) + v \\ & \sum_{j=1}^{n} s_j(A) s_j(A) s_j(B) - 2\sqrt{v}(\sum_{j=1}^{n} s_j^{\frac{1}{2}}(A) s_j^{\frac{3}{2}}(B))] \\ & + Kr' v^{2v} \sum_{j=1}^{n} [s_j(A^v B^{1-v})]^2 \\ & \geq v^2 [\|A\|_2^2 + \|B\|_2^2 - 2\|AB\|1] \\ & + r[] B\|_2^2 + v\|AB\|_1 - 2\sqrt{v}\sqrt{\|A\|_1}\sqrt{\|B^3\|_1} \\ & + Kr' v^{2v} \|A^v B^{1-v}\|_2^2 \end{aligned}$$

It is trivial that*:*

$$tr(\upsilon^2 A^2 + (1-v)^2 B^2) = \upsilon^2 trA^2 + (1-v)^2 trB^2 = v^2 \|A\|_2^2 + (1-v)^2 \|B\|_2^2$$

From two recent relations, it follows that:

$$\begin{aligned} & v^2 \|A\|_2^2 + (1-v)^2 \|B\|_2^2 \geq v^2 [\|A\|_2^2 + \|B\|_2^2 - 2\|AB\|_1] \\ & + r[\|B\|_2^2 + v\|AB\|_1 - 2\sqrt{v}\sqrt{\|A\|_1}\sqrt{\|B^3\|_1}] + Kr' v^{2v} \|A^v B^{1-v}\|_2^2 \end{aligned}$$

This estimate completes the proof of (3.7). The proof (3.8) is similar. So we omit its details. This completes the proof.

Conclusion

This paper obtained some refinements of Young type inequalities for scalars, then as applications of them, we presented some norm and trace inequalities.

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Authors Contributions

Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

Ethics

Interested readers are encouraged to find the applications for the derived scalar inequalities. This is an interesting topic for future research.

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