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q-Deformed Statistics from Position-Dependent Mass Schrödinger Equation

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Abstract: An algebraic approach is used to obtain the canonical form of the position-dependent mass Schrödinger equation from where a couple of canonical quantum variables, the *q*-deformed operators for the position x_q, and the hermitian linear momentum operator p_q are derived. In this *q*-deformed coordinate space, the commutator remains invariant namely $[x_q, p_q] = i\hbar$. By taking advantage of these *q*-deformed variables, one gets to a *q*-deformed exponential function $exp_q(x)$ as well as its corresponding *q*-deformed logarithm function $ln_q(x)$. From these *q*-deformed mathematical relations and from the fact that thermodynamic properties such as the internal energy *U*, entropy *S*, free energy *F*, heat capacity *C*, and others are related to the partition function *Z* and *ln*(*Z*), it is proposed their generalizations in terms of the *q*-deformed exponential and *q*-deformed logarithmic functions. As a result, the structure of Legendre transformations between these statistical properties remains invariant. The usefulness of the proposal is exemplified by considering two specific position-dependent mass distributions. In the same way, other possibilities could be used to generalize the statistical properties straightforwardly.

Keywords: Thermodynamic Properties, Deformed Exponential Function, Position-Dependent Mass

Introduction

Thermodynamic systems have been studied in the context of extensive and non-extensive properties through entropy or rather the mathematical properties of exponential and logarithmic functions with which the entropy is directly defined (Nivanen *et al*., 2003; Wang and Le Méhauté, 2002; Sargolzaeipor *et al*., 2018). Entropy is by construction, nonnegative, concave, and extensive function where the concept of extensive function in statistical mechanics refers to the fact that if *A* and *B* are two independent systems, then $p_{ij}(A+B) = p_i(A)p_j(B)$, such that $S(A+B) = S(A) + S(B)$ (Wang *et al*., 2002). This mathematical property can be achieved with the standard logarithm and exponential functions, namely $ln(AB) = ln(A) + ln(B)$ and $exp(A+B) = exp(A)exp(B)$. Unlike the extensive systems, the non-extensive ones obey the relationship of the type $S(A+B) = S(A) + S(B) + Q(q)S(A)S(B)$ from which, due to the factor $Q(q)S(A)S(B)$ we can identify a kind of pseudo-additivity that would be derived from a *q*-deformed exponential function involved in what has been called *q*-deformed algebra (Abe, 2001). In general, we would say that the *q*-algebras allow us to introduce the *q*-calculus in such a way that in the new coordinate space, we can solve the equation that describes the original problem or identify physical properties that from primitive space we could not solve. Specifically, the formalism of *q*-deformed algebra lies in the generalization of the standard exponential and logarithmic functions. From this generalization, it is possible to introduce a kind of *q*-deformed algebraic operations such as *q*-addition, *q*-subtraction, and *q*-product (Kaniadakis, 2001; 2002; 2005) as well as a *q*-deformed differential operator (*q*-calculus) generating a mathematical structure supported by a well-defined Abelian field (Scarfone, 2015). In this regard, the generalization of the statistics mechanics has been already demonstrated (Tsallis, 1988) based on the *q*-exponential function, preserving the main features of the ordinary Boltzmann-Gibbs statistical mechanics. Nowadays, several papers have been written on the foundations, the theoretical consistency, and the potential applications of the q-deformed exponential functions in statistical mechanics (Silva, 2006; Kim *et al*., 2019). Also, other specific applications have been considered including quantum

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entanglement (Ourabah *et al*., 2015), plasma physics (Lourek and Tribeche, 2016), genomics (Souza *et al*., 2014), and for predicting COVID-19 peaks (Tsallis and Tirnakli, 2020). Consequently, due to their multiple applications, the *q*-deformed exponential function has also been proposed in the treatment of the Position-Dependent MASS Schrödinger Equation (PDMSE), as an introduction to the concept of *q*-deformed quantum mechanics. Specifically, it has been related to a change in the linear momentum operator (Borges, 2004; Curado and Tsallis, 1991) which implies the existence of a relationship between the statistical mechanics and the *q*-deformed quantum mechanics. Indeed, it is well known that the displacement operator is directly related to the linear momentum operator through the exponential function (Costa Filho *et al*., 2011). For that, the linear momentum operator would be generalized through the *q*-deformed exponential function under the formalism of the *q*-algebras (da Costa *et al*., 2020). So, the *q*-deformed exponential function is related not only to the PDMSE for solving quantum interactions but also to the so-called *q*-statistics in the generalization of the additive property of the Boltzmann Gibbs (BG) entropy (Gomez and Borges, 2021). With this purpose, in the following, we begin by considering the *q*-deformed linear momentum operator in such a way that the canonical form of the position-dependent mass Schrödinger equation can be achieved. After that, it is presented the *q*-deformed quantum dynamic variables that are needed to obtain the generalization (*q*-deformed) of the most important thermodynamic properties. In the end, the usefulness of the proposed approach is exemplified by considering two different hyperbolic forms of position-dependent mass distributions.

The q-Deformed Quantum Mechanics

The q-Deformed Quantum Linear Momentum Operator

Starting with the position-dependent mass Schrödinger equation (Von Roos, 1983):

$$\widehat{K}_{\alpha\beta}\psi + V(x)\psi(u) = E\psi \tag{1}$$

where, the operator $\widehat{K}_{\alpha\beta}$ is the von Ross kinetic energy operator given by:

$$\widehat{K}\alpha\beta = -\frac{h^2}{4}\left(m^{\alpha}(q;x)\frac{d}{dx}m^{\beta}(q;x)\frac{d}{dx}m^{\gamma}(q;x) + m^{\gamma}(q;x)\frac{d}{dx}m^{\beta}(q;x)\frac{d}{dx}m^{\alpha}(q;x)\right) \tag{2}$$

with, $m(q; x)$ the mass distribution, q is the mass parameter and the ambiguity parameters fulfill the restriction $\alpha + \beta + \gamma = -1$. By applying the derivative operator on the mass distribution, it can be rewritten as in Rego-Monteiro *et al*. (2016) given:

$$\left[-\frac{d}{dx}\left(\frac{\hbar}{2m(q;x)}\right)\frac{d}{dx} + \cup_{\alpha\beta}(x)\right]\psi(x) = E\psi(x) \tag{3}$$

where:

$$U_{\alpha\beta}(x) = V(x) + \frac{\hbar^2}{4}(\beta+1)\frac{m''(q;x)}{m^2(q;x)}$$
$$U_{\alpha\beta}(x) = V(x) + \frac{\hbar^2}{4}(\beta+1)\frac{m''(q;x)}{m^2(q;x)} \tag{4}$$

With the aim of transforming Eq. (3) into its canonical form, we factorize the Hamiltonian of Eq. (3) as follows:

$$\hat{H} = \frac{d}{dx}\left(\frac{i\hbar}{\sqrt{2m(q;x)}}\right)\left(\frac{i\hbar}{\sqrt{2m(q;x)}}\right)\frac{d}{dx} + U_{\alpha\beta}(x) \tag{5}$$

such that if we use the commutator:

$$\left[\frac{d}{dx}, \quad \frac{i\hbar}{\sqrt{2m(q;x)}}\right] = \left(\frac{1}{\sqrt{2m(q;x)}}\right)' \tag{6}$$

we have:

$$\hat{H} = \left[\frac{i\hbar}{\sqrt{2m(q;x)}}\frac{d}{dx} + i\hbar\left(\frac{1}{\sqrt{2m(q;x)}}\right)'\right]\left[\frac{i\hbar}{\sqrt{2m(q;x)}}\frac{d}{dx}\right] + U_{\alpha\beta}(x) \tag{7}$$

Thus, we can write:

$$\hat{H} = \left(\frac{i\hbar}{\sqrt{2m(q;x)}}\frac{d}{dx}\right)^2 - \frac{i\hbar m'(q;x)}{4m(q;x)\sqrt{2m(q;x)}}\left(\frac{i\hbar}{\sqrt{2m(q;x)}}\frac{d}{dx}\right) - \left(\frac{i\hbar}{\sqrt{2m(q;x)}}\frac{d}{dx}\right)\left(\frac{i\hbar m'(q;x)}{4m(q;x)\sqrt{2m(q;x)}}\right) - \frac{\hbar^2\left(\frac{m'(q;x)}{4m(q;x)\sqrt{m(q;x)}}\right)'}{\sqrt{2m(q;x)}} + U_{\alpha\beta}(x) \tag{8}$$

where the commutator has been used:

$$\left[\frac{i\hbar}{\sqrt{2m(q;x)}}\frac{d}{dx}, \frac{i\hbar m'(q;x)}{4m(q;x)\sqrt{2m(q;x)}}\right] = \frac{i\hbar}{\sqrt{2m(q;x)}}\left(\frac{m'(q;x)}{4m(q;x)\sqrt{m(q;x)}}\right)' \tag{9}$$

At this point, it should be noticed that the apostrophe refers to a standard derivative with respect to the position. Finally, we have the canonical form of the Hamiltonian:

$$\widehat{H} = \frac{1}{2m_0}\hat{p}_q^2 + u_{eff}(x) \tag{10}$$

where:

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$$\hat{p}_q = -\frac{i\hbar}{\sqrt{M(q;x)}}\frac{d}{dx} - \frac{i\hbar}{2}\left(\frac{1}{\sqrt{M(q;x)}}\right)' \quad (11)$$

is the q-deformed position-dependent mass linear momentum operator, $M(q;x) = m(q;x)/m_0$ and $u_{eff}(x)$ is the effective potential:

$$u_{eff}(x) = \frac{\hbar^2}{2m_0}\left[\left(\left(\frac{1}{2\sqrt{M(q;x)}}\right)'\right)^2 + \frac{\hbar^2}{\sqrt{M(q;x)}}\left(\frac{1}{2\sqrt{M(q;x)}}\right)''\right] + U_{\alpha\beta}(x) \quad (12)$$

that, after using the potential $U_{\alpha\beta}(x)$ given in Eq. (4) leads to:

$$u_{eff}(x) = V(x) + \frac{\hbar^2}{4m_0}\left(\beta + \frac{1}{2}\right)\frac{M''(q;x)}{M^2(q;x)} - \frac{\hbar^2}{2m_0}\left[\alpha(\alpha+\beta+1) + \beta + \frac{9}{16}\right]\frac{\left(M'(q;x)\right)^2}{M^3(q;x)} \quad (13)$$

The particular case of constant mass $m(0;x) = m_0$ gives place to the standard operators:

$$\hat{p}_0 = \hat{p} = -i\hbar d/dx \text{ and } u_{eff}(x) = V(x) \quad (14)$$

It is worth mentioning that the generalized linear momentum operator $\hat{p}_q$ given in Eq. (11) is a Hermitian operator. In fact, any operator of the form:

$$\hat{A} = -ig(x)\frac{d}{dx} - if(x) \quad (15)$$

fulfill the condition:

$$\int (\hat{A}\psi)^*\psi dx) = \int \psi^* \hat{A}\psi\, dx) + i\int \psi^*(2f(x) - g'(x))\psi\, dx \quad (16)$$

Here, $\hat{p}_q = \hat{A}$ on condition to have $g(x) = \frac{\hbar}{\sqrt{M(q;x)}}$ $f(x) = \frac{1}{2}\left(\frac{\hbar}{\sqrt{M(q;x)}}\right)'$. In such case the second term of Eq. (16) banishes and we have:

$$\int (\hat{p}_q\psi)^*\psi dx) = \int \psi^* \hat{p}_q\psi dx) \quad (17)$$

showing that the operator $\hat{p}_q$ is a hermitian operator and consequently the Hamiltonian operator $\hat{H}$ given in Eq. (10) is also Hermitian, which is a sufficient condition to deal with real eigenvalues. Additionally, the operator $\hat{p}_q$ could not be self-adjoint if the $(\hat{p}_q, D) \neq (\hat{p}_q^{\dagger}, D')$ inequality holds. In that case, we would be dealing with a self-adjoint extension for the operator $(\hat{p}_q, D)$ (Gadella *et al*., 2007). On the other hand, if the domains D and D' match then the $\hat{p}_q$ operator could be self-adjoint. This latter property is also determined by the mass distribution $M(q;x)$.

Canonical Transformation

To solve the canonical Schrödinger equation $\hat{H}\psi = E\psi$ with $\hat{H}$ given in Eq. (10) for some interaction potential V and mass distribution $m(q;x) = m_0 M(q;x)$, we propose the point canonical transformation:

$$x_q = \int \sqrt{M(q;x)}dx \quad (18)$$

leading to:

$$-\frac{\hbar^2}{2m_0}\frac{d^2\psi}{dx_q^2} + \frac{\hbar^2}{2m_0}\left(ln\sqrt{m(q;x_q)}\right)'\frac{d\psi}{dx_q} + \left[V + \frac{\hbar^2}{2m_0}(\beta+1)\left(ln\sqrt{m(q;x_q)}\right)'' - \frac{\hbar^2}{2m_0}(4\alpha(\alpha+\beta+1)+\beta+1)\left(\left(ln\sqrt{m(q;x_q)}\right)'\right)^2\right]\psi = E\psi \quad (19)$$

Thus, by applying the similarity transformation:

$$\psi = \left(m(q;x)\right)^{1/4}\phi(x_q) \quad (20)$$

we have:

$$-\frac{\hbar}{2m_0}\frac{d^2\phi}{dx_q^2} + U\phi = E\phi \quad (21)$$

where:

$$U = V + \frac{\hbar^2}{2m_0}\left(\beta + \frac{1}{2}\right)\left(ln\sqrt{m(q;x)}\right)'' - \frac{\hbar}{2m_0}\left(4\alpha(\alpha+\beta+1) + \beta + \frac{3}{4}\right)\left(\left(ln\sqrt{m(q;x)}\right)'\right)^2 \quad (22)$$

The Schrödinger equation given above has been solved for different mass distributions under different interaction potentials, so in this study, we will only focus on using the transformation derived in the previous formalism to extend its applications in the q-deformed thermodynamic properties.

Generalization (q-Deformed) of Thermodynamic Properties

The q-Deformed Quantum Dynamic Variables

The generalized quantum dynamic variables, namely the q-deformed linear momentum operator p_q and the

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canonical transformation x_q given in Eqs. (11) and (18) respectively, preserve invariant the quantum commutation relationship, namely $\left[x_q, p_q\right] = i\hbar$ Explicitly:

$$\begin{aligned}&\left[\int \sqrt{M(q;x)}\,dx\ ,\ -\frac{i\hbar}{\sqrt{M(q;x)}}\frac{d}{dx}-\frac{i\hbar}{2}\left(\frac{1}{\sqrt{M(q;x)}}\right)'\right]=\\&-\left(\int \sqrt{M(q;x)}\,dx\right)\left(-\frac{i\hbar}{\sqrt{M(q;x)}}\frac{d}{dx}\right)-\left(\int \sqrt{M(q;x)}\,dx\right)\left(\frac{i\hbar}{2}\left(\frac{1}{\sqrt{M(q;x)}}\right)'\right)\\&+\frac{i\hbar}{\sqrt{M(q;x)}}\frac{d}{dx}\left(\int \sqrt{M(q;x)}\,dx\right)+\frac{i\hbar}{2}\left(\frac{1}{\sqrt{M(q;x)}}\right)'\left(\int \sqrt{M(q;x)}\,dx\right)\\&=\frac{i\hbar}{\sqrt{M(q;x)}}\frac{d\int \sqrt{M(q;x)}\,dx}{dx}=i\hbar\end{aligned} \tag{23}$$

Furthermore, through these new quantum dynamic variables, we can introduce the generalized (*q*-deformed) exponential function:

$$exp_q(x) = exp(x_q) \tag{24}$$

such that:

$$\lim_{q\to 0} exp_q(x) = exp(x) \tag{25}$$

being $exp(x)$ the standard exponential function.

In addition, if the transformation given in Eq. (18) has an inverse, the generalized (*q*-deformed) logarithmic function will be:

$$ln_q(x) = x_q^{-1}\left(ln(x)\right) \tag{26}$$

Generalized (q-Deformed) Statistic Properties

It is well known that statistical properties such as the internal energy *U*, entropy *S*, free energy *F*, and heat capacity *C* are defined through the partition function *Z*(*T*) and its logarithm *ln*(*Z*) (Peña *et al*., 2016). It is defined as:

$$Z = \sum_i^{\Omega} exp\left(-\in_i /kT\right) \tag{27}$$

where, Ω is the total number of allowed states of the system with probabilities given by the Boltzmann distribution (Tsallis, 2009):

$$p_i = \frac{1}{Z} exp\left(-\in_i /kT\right) \tag{28}$$

on condition that, $\sum_i^{\Omega} p_i = 1$

Thus, the internal energy *U* comes from:

$$U = \sum_i^{\Omega} p_i \in_i = -\frac{\partial}{\partial\beta} lnZ \tag{29}$$

At this point, it is worth mentioning that the *q*-deformed exponential function exp_q (*x*) given in Eq. (24) and the *q*-deformed logarithm function of Eq. (26) can be used to introduce a generalized (*q*-deformed) partition function Z_q and consequently the internal energy U_q. Namely:

$$Z_q = \sum_i^{\Omega} exp_q(-\beta \in_i) \tag{30}$$

and:

$$U_q = -\frac{\partial}{\partial\beta} ln(Z_q) \tag{31}$$

with $\beta = \frac{1}{kT}$. Thence, due to the fact that the partition function *Z* and the internal energy *U* are involved with other statistic potentials (Peña *et al*., 2016) through the so-called Legendre transformations, namely the Entropy $S = k\,ln(Z) + k\beta U$, the Helmholtz free energy $F = -\frac{1}{\beta} ln(Z)$, and the heat capacity $C = -k\beta^2 \frac{\partial U}{\partial \beta}$, by preserving the structure of the Legendre transformations, it is possible to get their corresponding generalized expressions as follow: $S_q = k\,ln_q(Z_q) + k\beta U_q$ $F_q = -\frac{1}{\beta} ln_q(Z_q)$ and $C_q = -k\beta^2 \frac{dU_q}{d\beta}$ and. The next section will give some explicit examples.

Application to Thermodynamic Properties

This section is devoted to showing the usefulness of the proposal by considering two different position-dependent mass distributions of hyperbolic type.

Mass Distribution $m(q; x)=m_0\, cosh^2(qx)$

In this case $M(q; x) = \cosh^2 (qx)$ such that, from Eqs. (11) and (18) The *q*-deformed quantum dynamic variables are:

$$\hat{p}_q = -i\hbar sech(qx)\frac{d}{dx} + i\hbar\frac{q}{2} sech(qx) tanh(qx) \tag{32}$$

and:

$$x_q = \frac{1}{q} sinh(qx) \tag{33}$$

Straightforwardly, for this case, the quantum commutation relation between x_q and p_q remains unchanged.

Furthermore, in accordance with Eq. (24), the generalized *q*-deformed exponential function is:

$$exp_q(x) = exp\left(\frac{1}{q} sinh(qx)\right) \tag{34}$$

whose partner inverse function is defined as the generalized q-deformed logarithmic relationship:

$$ln_q(x) = \frac{1}{q} sinh^{-1}\left(qln(x)\right) \tag{35}$$

So, by using the identity:

$$sinh^{-1}(x) = ln\left(x + \sqrt{x^2 + 1}\right) \tag{36}$$

one gets:

$$ln_q(x) = ln\left[qln(x) + \sqrt{1 + q^2 ln^2(x)}\right]^{\frac{1}{q}} \tag{37}$$

Also, it is easily observed that:

$$\lim_{q\to 0} exp_q(x) = exp(x) \text{ and } \lim_{q\to 0} ln_q(x) = ln(x) \tag{38}$$

Consequently, from the Eqs. (30) and (34) we can write the q-deformed partition function Z_q (T) as:

$$Z_q = \sum_i^{\Omega} exp\left(\frac{-sinh(q\beta \in_i)}{q}\right) \tag{39}$$

Hence, the generalized (q-deformed) internal energy comes from the Eq. (31) as:

$$U_q = -\frac{\partial}{\partial\beta} ln\left(q\, ln(Z_q) + \sqrt{1 + q^2 ln^2(Z_q)}\right)^{\frac{1}{q}} \tag{40}$$

Also, in view of Eq. (38), one has:

$$\lim_{q\to 0} Z_q = Z \tag{41}$$

and:

$$\lim_{q\to 0} U_q = -\frac{\partial}{\partial\beta} ln(Z_q) = U \tag{42}$$

Finally, by following the structure of the Legendre transformation among some thermodynamic functions, the generalization of the internal energy given in Eq. (31) and its related functions are rewritten as follows:

$$S_q = k\, ln_q(Z_q) - k\beta U_q \tag{43}$$

where:

$$U_q = \frac{\partial}{\partial\beta} ln\left(q\, ln(Z_q) + \sqrt{1 + q^2 ln^2(Z_q)}\right)^{\frac{1}{q}},$$
$$F_q = -\frac{1}{\beta} ln\left(q\, ln(Z_q) + \sqrt{1 + q^2 ln^2(Z_q)}\right)^{\frac{1}{q}} \tag{44}$$

and:

$$C_q = -k\beta^2 \frac{\partial U_q}{2\beta} = -k\beta^2 \frac{\partial}{\partial\beta}\left(-\frac{\partial}{\partial\beta} ln_q(Z_q)\right) =$$
$$k\beta^2 \frac{\partial^2}{\partial\beta^2} ln\left(q\, ln(Z_q) + \sqrt{1 + q^2 ln^2(Z_q)}\right)^{\frac{1}{q}}. \tag{45}$$

As before, when the q parameter tends to zero, all the above-generalized statistic properties reduce to their corresponding standard ones.

Mass Distribution $m(q; x)=m_0\, cosh^4(qx)$

In this new situation, from Eqs. (11) and (18) the q-deformed quantum variables are:

$$\hat{p}_q = -i\hbar cosh^2(qx)\frac{d}{dx} - i\hbar q cosh(qx) sinh(qx) \tag{46}$$

and:

$$x_q = \frac{1}{q} tanh(qx) \tag{47}$$

fulfilling the commutation relationship $[x_q, p_q] = ih$

Likewise, from Eq. (24), the generalized q-deformed exponential function results in:

$$exp_q(x) = exp\left(\frac{1}{q} tanh(qx)\right) \tag{48}$$

and the corresponding inverse function is:

$$ln_q(x) = \frac{1}{q} tanh^{-1}(qln(x)) \tag{49}$$

which, after using the identity:

$$tanh^{-1}(x) = \frac{1}{2} ln\left(\frac{1+x}{1-x}\right) \tag{50}$$

we have:

$$ln_q(x) = \frac{1}{2q} ln\left(\frac{1 + qln(x)}{1 - qln(x)}\right) \tag{51}$$

Also, the $q \longrightarrow 0$ limit leads to:

$$\lim_{q\to 0} exp_q(x) = exp(x) \tag{52}$$

and:

$$\lim_{q\to 0} ln_q(x) = ln(x) \tag{53}$$

Eq. (30) together with Eq. (48) leads to the q-deformed partition function:

$$Z_q = \sum_{i}^{\Omega} exp\left(\frac{1}{q} tanh(-q\beta\epsilon_i)\right) \quad (54)$$

Hence, the generalized (q-deformed) internal energy comes from the Eq. (31) as:

$$U_q = -\frac{\partial}{\partial\beta} ln\left(\frac{1+qln(Z_q)}{1-qln(Z_q)}\right)^{\frac{1}{2q}} \quad (55)$$

By virtue of the result given in Eqs. (52-53) we have:

$$\lim_{q\to 0} Z_q = Z \quad (56)$$

as well as:

$$\lim_{q\to 0} U_q = U \quad (57)$$

Finally, the Legendre transformations of the thermodynamic functions are generalized as follows:

$$S_q = k\, ln_q(z_q) - k\beta U_q \quad (58)$$

where:

$$U_q = \frac{1}{2q}\frac{\partial}{\partial\beta} ln\left(\frac{1+qln(Z_q)}{1-qln(Z_q)}\right) \quad (59)$$

$$F_q = -\frac{1}{\beta} ln_q(Z_q) = -\frac{1}{2\beta q} ln\left(\frac{1+qln(Z_q)}{1-qln(Z_q)}\right) \quad (60)$$

and:

$$C_q = -k\beta^2 \frac{\partial U_q}{\partial\beta} = \frac{k\beta^2}{2q}\frac{\partial^2}{\partial\beta^2} ln\left(\frac{1+qln(Z_q)}{1-qln(Z_q)}\right) \quad (61)$$

As expected, from expressions given in Eqs. (56-57) the limit case $q \to 0$ leads to the standard statistic potentials, namely:

$$\lim_{q\to 0} S_q = S, \lim_{q\to 0} F_q = F \; and \lim_{q\to 0} C_q = C \quad (62)$$

Recovering the standard expressions for the statistic properties entropy S, Helmholtz free energy F and the heat capacity C.

Materials and Methods

This study is theoretical research on the field of Quantum mechanics. Specifically, on the q-deformed form for which the study begins with the search of the q-deformed Quantum linear momentum operator. From there, the q-deformed exponential function $exp_q(x) = exp(x_q)$ as well as the corresponding q-deformed logarithm function $ln_q(x) = x_q^{-1}(ln\,(x))$, which come from the canonical form of the PDM Schrödinger equation, were used to generalize the thermodynamic potentials defined through their corresponding standard functions. With these elements, the hyperbolic mass distributions such as $m(q;x) = m_0 cosh^2(qx)$ and $m(q;x) = m_0 cosh^4(qx)$ were used for exemplifying the proposal.

Results and Discussion

From the hyperbolic mass distribution $m(q;x) = m_0 cosh^2(qx)$, the q-exponential function:

$$exp_q(x) = exp\left(\frac{1}{q} sinh(qx)\right)$$

and the q-logarithm function:

$$ln_q(x) = ln\left[qln(x) + \sqrt{1+q^2ln^2(x)}\right]^{\frac{1}{q}}$$

were obtained.

With these results, the corresponding generalized thermodynamic properties become:

$$U_q = -\frac{\partial}{\partial\beta} ln\left(q\, ln(Z_q) + \sqrt{1+q^2ln^2(Z_q)}\right)^{\frac{1}{q}}$$

$$S_q = k\, ln_q(Z_q) - k\beta U_q$$

$$F_q = -\frac{1}{\beta} ln\left(q\, ln(Z_q) + \sqrt{1+q^2ln^2(Z_q)}\right)^{\frac{1}{q}}$$

and:

$$C_q = k\beta^2 \frac{\partial^2}{\partial\beta^2} ln\left(q\, ln(Z_q) + \sqrt{1+q^2ln^2(Z_q)}\right)^{\frac{1}{q}}$$

Likewise, when we use the mass distribution $m(q;x) = m_0 cosh^4(qx)$, the q-exponential function and the q-logarithm function are respectively given by:

$$exp_q(x) = exp\left(\frac{1}{q} tanh(qx)\right)$$

and:

$$ln_q(x) = \frac{1}{2q} ln\left(\frac{1+qln(x)}{1-qln(x)}\right)$$

Consequently, the corresponding generalized thermodynamic properties:

$$U_q = \frac{1}{2q}\frac{\partial}{\partial\beta}ln\left(\frac{1+qln(Z_q)}{1-qln(Z_q)}\right)$$

$$F_q = -\frac{1}{\beta}ln_q(Z_q) = -\frac{1}{2\beta q}ln\left(\frac{1+qln(Z_q)}{1-qln(Z_q)}\right)$$

and:

$$C_q = -k\beta^2\frac{\partial U_q}{\partial\beta} = \frac{k\beta^2}{2q}\frac{\partial^2}{\partial\beta^2}ln\left(\frac{1+qln(Z_q)}{1-qln(Z_q)}\right)$$

were derived.

Conclusion

Based on the *q*-deformed quantum variables *xq* and *pq* deduced from a purely algebraic approach, the purpose of this study has been to propose a generalization of the partition function and the internal energy through a *q*-deformed exponential function and its partner *q*-deformed logarithmic function. In addition, taking advantage of the above approach, some thermodynamic properties lie in the partition function and the internal energy, which have also been generalized. These generalizations are straightforward since such potentials are given in terms of the Legendre relations, which remain invariant under this treatment. Namely, from the *q*-deformed generalized partition unction $Z_q = \sum_i^\Omega exp_q\left(-\beta \in_i\right)$ all the other related thermodynamic functions are directly generalized. In order to show the usefulness of our proposal, we have considered two hyperbolic position-dependent mass distributions although the method is general and can be adapted to other situations.

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Author’s Contributions

Jesus Morales Rivas: Supervision, read, reviewed and edited of the final version.

Jose Juan Peña Gil: Methodology, mathematical calculations, visualization and written drafted.

J. García Ravelo: Conceptualization, analysis and validation.

Ethics

This is an original mathematical article and no ethical issues can arise after its publication in line with university and international standards.

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