

Asymptotic Analysis and Convergence of Recursively Defined Sequences

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Abstract: The asymptotic properties and limiting behavior of several distinct real sequences are determined. These sequences are defined by means of recurrence relations. Divergent and convergent sequences are discussed. It is shown methods can be found to treat both kinds producing definite knowledge with regard to the sequence. In the case of divergence, a related convergent subsequence can often be found by examining the asymptotic results. Although the methods may have only a limited applicability, some theorems are given to give more of a synthesis to the work.

Keywords: Sequence, Asymptotic, Limit, Iteration

Introduction

Infinite sequences which are defined by recursive formulas such as recursion relations often have interesting behavioral properties, admit asymptotic formulas and approach a limit which can be calculated. The idea is to introduce a variety of approaches and algorithms which can be applied to study sequences. These include study of large n behavior, determining bounds and whether a limit exists can be calculated (Stromberg, 1981; Bromwich, 1926; Knopp, 1948). These procedures may be case specific and may work over a large number of examples, or may not apply at all. Often the exact approach depends on the particular form of the actual recursion. Some theorems are also presented which rigorously cover large classes of sequences which perhaps satisfy assumptions and give a sense of synthesis.

It is often the case that sequences have applications to a many areas such as the study of dynamical systems and chaos (Bai-Lin, 1984; Devaney, 1986; Polya and Szego, 1972) and discretized equations. Some approaches may work quite well for certain classes of sequence, but fail for others, so an absence of universality of the approach may be absent. The paper consists of a number of sections closely linked. The first four sections concentrate on specific sequences and their behavior. The last six present more general theorems with proofs, which apply to any sequence that satisfies the conditions of the theorem. As far as novelty is concerned, the estimates for the first three sequences and the geometric problem are new, as are some of the theorems. The

methods and ideas introduced here have wide ranging applications in many areas of real and complex analysis (Whittaker and Watson, 2020; Furdui, 2013; Berndt, 2012; Bourchtein and Bourchtein, 2022).

The Sequence $x_{n+1} = x_n + \frac{1}{x_n}$

The first sequence studied is defined for $n \in \mathbb{N}$ by setting $x_1 = 1$ and for $n > 1$

$$x_{n+1} = x_n + \frac{1}{x_n}, \quad (1)$$

The sequence x_n can be estimated in such a way that the dependence on n is obtained. Let us start with an approach which starts by squaring both sides of (1)

$$x_m^2 = x_{m-1}^2 + \frac{1}{x_{m-1}^2} + 2. \quad (2)$$

Sum on both sides of (2) from $m = 2$ to some integer $m = n$ and use $x_1 = 1$ to yield x_n^2 in the form

$$x_n^2 = x_1^2 + \frac{1}{x_1^2} + \frac{1}{x_2^2} + \cdots + \frac{1}{x_{n-1}^2} + 2(n-1) = 2n + \frac{1}{x_2^2} + \cdots + \frac{1}{x_{n-1}^2}. \quad (3)$$

For $n = 2$, we have $x_2^2 = 4$ and for $n \geq 3$, there is the following lower bound for x_n

$$x_n^2 > 2n. \quad (4)$$

Use the upper bound given by putting (4) in (3) to produce an upper bound for x_n^2

$$x_n^2 < 2n + \frac{1}{2 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{2 \cdot (n-1)} = 2n + \frac{1}{2} \left(\frac{1}{2} + \cdots + \frac{1}{n-1} \right). \quad (5)$$

The harmonic series, $H_{n-1} = \sum_{k=1}^{n-1} 1/k$, appears in the bracket in (5). For this result, there is an upper bound which depends on Euler's constant γ given by,

$$H_n = \sum_{k=1}^n \frac{1}{k} < \log n + \gamma + \frac{1}{2n}.$$

Then (5) gives the following upper bound for x_n^2 ,

$$x_n^2 < 2n + \frac{1}{2} \log n + \frac{1}{2}(\gamma - 1) + \frac{1}{4n}. \quad (6)$$

Using (6) in (3), with $C = \gamma - 1$, a lower bound for x_n^2 results in the form

$$x_n^2 > 2n + \frac{1}{2} \cdot \left(\frac{1}{\frac{1}{2} + \frac{1}{4} \log 2 + \frac{C}{2} + \frac{1}{8}} + \frac{1}{3 + \frac{1}{4} \log 3 + \frac{C}{2} + \frac{1}{8}} + \dots + \frac{1}{n-1 + \frac{1}{4} \log(n-1) + \frac{C}{2} + \frac{1}{8}} \right). \quad (7)$$

Two bounds have been obtained, an upper bound (6) and a lower bound (7). Now add and subtract the terms $(\log n)/2$ and $H_{n-1}/2$ to the right side of (7). A lower bound for x_n^2 results

$$x_n^2 > 2n + \frac{1}{2} \log n + \frac{1}{2} \left(\sum_{k=1}^{n-1} \frac{1}{k} - \log n \right) - \frac{1}{2} - \frac{1}{8} \sum_{k=2}^{n-1} \frac{\log k + 2C + \frac{1}{2k}}{k(k + \frac{1}{4} \log k + \frac{C}{2} + \frac{1}{8k})}. \quad (8)$$

The last term in series (8) converges by comparison with the series $\sum \log k/k^2$ and $\sum 1/k^2$.

Define a function $G(n)$ which is defined using (3) and for $n \in \mathbb{N}$ with $n \geq 3$ to be

$$G(n) = 2n + \frac{1}{2} \log n - x_n^2 = 2n + \frac{1}{2} \log n - 2n - \sum_{k=2}^{n-1} \frac{1}{x_k^2}.$$

Adding $(-H_{n-1} + H_{n-1})/2$ to the right,

$$\begin{aligned} &= \frac{1}{2}(\log n - H_n) + \frac{1}{2n} + \sum_{k=1}^{n-1} \frac{1}{2k} - \sum_{k=2}^{n-1} \frac{1}{x_k^2} \\ &= \frac{1}{2} \cdot (\log n - H_n) + \frac{1}{2n} + \frac{1}{2} + \frac{1}{4} - \frac{1}{4} + \sum_{k=3}^{n-1} \left(\frac{1}{2k} - \frac{1}{x_k^2} \right) \\ &= \frac{1}{2} - \frac{1}{2} \cdot \left(\sum_{k=1}^n \frac{1}{k} - \log n \right) + \frac{1}{2n} + \sum_{k=3}^{n-1} \left(\frac{1}{2k} - \frac{1}{x_k^2} \right). \end{aligned} \quad (9)$$

For all $k \geq 3$, the following lower bound holds

$$\frac{1}{2k} - \frac{1}{x_k^2} > 0.$$

From this inequality we conclude that the sequence S_{n-1} defined as

$$S_{n-1} = \sum_{k=3}^{n-1} \left(\frac{1}{2k} - \frac{1}{x_k^2} \right) \quad (10)$$

and increases monotonically with n . The sequence S_{n-1} is bounded above as can be seen by dropping the last two terms in (6). This suffices to calculate the estimate

$$\frac{1}{x_k^2} > \frac{1}{2k + \frac{1}{2} \log k}.$$

Using this bound, this implies that

$$\begin{aligned} S_n &= \sum_{k=3}^M \left(\frac{1}{2k} - \frac{1}{x_k^2} \right) < \sum_{k=3}^M \left(\frac{1}{2k} - \frac{1}{2k + \frac{1}{2} \log k} \right) = \frac{1}{8} \cdot \\ &\sum_{k=3}^{\infty} \frac{\log k}{k(k + \frac{1}{2} \log k)} \quad (11) \\ &< \frac{1}{8} \sum_{k=3}^M \frac{\log k}{k^2} < \frac{1}{8} \sum_{k=3}^{\infty} \frac{\log k}{k^2}. \end{aligned}$$

Since S_n is bounded above and increasing, it converges to a finite limit called S by the monotone convergence theorem. Let γ denote Euler's constant, then the sequence

$$\frac{\gamma}{2} < \frac{1}{2} \left(\sum_{k=1}^n \frac{1}{k} - \log n \right) < \frac{1}{2} \quad (12)$$

is monotonically decreasing and approaches $\gamma/2$. The two sequences (10) and (12) appear in the function $G(n)$. Therefore, from (9), it follows that the sequence $G(n)$ converges to a finite limit as $n \rightarrow \infty$. If the limit of (9) is called G , then based on (12) we have

$$G = \frac{1}{2} - \frac{\gamma}{2} + S. \quad (13)$$

It is also worth pointing out that the function $G(n)$ also satisfies the following bounds, using (8) on the right,

$$\begin{aligned} &\frac{1}{2} - \frac{1}{2} \left(\sum_{k=1}^n \frac{1}{k} - \log n \right) + \frac{1}{2n} < G(n) \\ &< \frac{1}{2} - \frac{1}{2} \left(\sum_{k=1}^{n-1} \frac{1}{k} - \log n \right) + \frac{1}{8} \sum_{k=2}^{n-1} \frac{\log k + 2C + \frac{1}{2k}}{k(k + \frac{1}{4} \log k + \frac{C}{2} + \frac{1}{8k})}. \end{aligned}$$

Some further properties of the sequence $G(n)$ ⁽¹⁴⁾ can be deduced from these results. From the definition of $G(n)$ in (9), it follows that $G(1) = 1$. The inequality on the right of (14) implies that $G(n) < 1$ for all $n > 1$, and hence $\sup G(n) = \max G(n) = 1$. The value of $\inf G(n)$ can be determined. By calculation the function $G(n)$ is monotonically decreasing until $n = 13$ at which point it increases monotonically. This means that $\inf G(n) = \min G(n) = G(13)$. This can be shown by calculating numerically from $n = 2$ to 12 and the result is $G(n) > G(n+1)$. This can be continued to where the following bounds can be used. Starting with the result

$$G(n) - G(n+1) = \frac{1}{x_n^2} - \frac{1}{2} \log \left(1 + \frac{1}{n} \right), \quad (15)$$

introduce sequences γ_n and α_n defined by

$$\begin{aligned} \gamma_{n-1} &= \sum_{k=1}^{n-1} \frac{1}{k} - \log n, & \alpha_{n-1} &= \\ &\sum_{k=2}^{n-1} \frac{1}{k(k + \frac{1}{4} \log k + \frac{C}{2} + \frac{1}{8k})}. \end{aligned}$$

Use the inequality $\log(1 + \frac{1}{n}) > \frac{1}{n} - \frac{1}{2n^2}$, to produce the following upper bound

$$\begin{aligned} &\frac{1}{x_n^2} - \frac{1}{2} \log \left(1 + \frac{1}{n} \right) \\ &< \frac{-\frac{1}{2} \log n + \frac{1}{2} - \frac{1}{2} \gamma_{n-1} + \frac{1}{8} \alpha_{n-1} + 1 + \frac{1}{4n} (\log n - 1) + \frac{\gamma_{n-1}}{4n} - \frac{\alpha_{n-1}}{16n}}{4n^2 (1 + \frac{1}{4n} (\log n - 1) + \frac{\gamma_{n-1}}{4n} - \frac{\alpha_{n-1}}{16n})}. \end{aligned} \quad (16)$$

The right hand side of (16) becomes negative after some value $n = n_0$ at which point the term $\log n$ dominates the expression since the denominator is positive.

The Sequence $x_n = x_{n-1} - x_{n-1}^2$

A second sequence is defined by picking an initial value x_0 in the interval $0 < x_0 < 1$. The sequence terms x_n for $n \in \mathbb{N}$ are defined as $x_n = x_{n-1} - x_{n-1}^2$. In terms of the x_n a more complicated sequence can be defined for $n \geq 2$ by

$$s_n = \frac{n(1-x_n)}{\log n}. \quad (17)$$

The asymptotic behavior of this sequence and the limit will be studied. To begin transform this sequence into a related sequence by using the transformation $y_n = 1/x_n$ where x_0 maps into $1 < y_0 < \infty$ and

$$\frac{1}{y_{n+1}} = \frac{1}{y_n} - \frac{1}{y_n^2} = \frac{y_n - 1}{y_n^2}. \quad (18)$$

Solving (18) for y_{n+1} , we obtain

$$y_{n+1} = \frac{y_n^2}{y_n - 1} = y_n + 1 + \frac{1}{y_n - 1}. \quad (19)$$

Since $x_{n+1} = x_n - x_n^2 < x_n$, sequence x_n decreases monotonically, which implies that $y_{n+1} > y_n$. Hence, sequence y_n increases monotonically such that $y_n > 1$. To improve the estimate for y_n , proceed inductively using the following approach. The first step is to note $y_1 > y_0 + 1 + 1/(y_0 - 1) > 2$, hence inductively if $y_{n-1} > n$, we have

$$y_n = y_{n-1} + 1 + \frac{1}{y_{n-1} - 1} > n + 1 + \frac{1}{y_{n-1} - 1} > n + 1. \quad (20)$$

To get an upper bound for y_n , consider (19) and use (20), repeatedly

$$\begin{aligned} y_n &= y_{n-1} + 1 + \frac{1}{y_{n-1} - 1} \leq y_{n-1} + 1 + \frac{1}{n+1} \\ &\leq y_{n-2} + 2 + \frac{1}{n-1} + \frac{1}{n-2} \leq \dots \leq y_1 + n - 1 + \sum_{m=1}^{n-1} \frac{1}{m}. \end{aligned} \quad (21)$$

With γ is Euler's constant as usual, the harmonic series H_n which appears in (21) has the upper bound

$$H_{n-1} = \sum_{m=1}^{n-1} \frac{1}{m} \leq \log n + \gamma - \frac{1}{2n}.$$

Define the constant $A(y_1) = y_1 - 1 + \gamma > 0$, so the upper bound can be written

$$y_n < n + \log n + A(y_1) - \frac{1}{2n}. \quad (22)$$

Now using the upper bound for y_n in (22), a lower bound for $1/(y_n - 1)$ is obtained,

$$\frac{1}{y_n - 1} \geq \frac{1}{n - 1 + \log n + A(y_1) - \frac{1}{2n}}. \quad (23)$$

Putting this in (19) and then iterating it until the term y_1 is reached, we see that

$$y_n = y_{n-1} + 1 + \frac{1}{y_{n-1} - 1} \geq y_{n-1} + 1 + \frac{1}{n - 2 + \log(n-1) + A}$$

$$\begin{aligned} &\geq y_{n-2} + 2 + \frac{1}{n-3+\log(n-2)+A(y_1)} + \frac{1}{n-2+\log(n-1)+A(y_1)} \geq \\ &\dots \geq y_1 + n - 1 + \sum_{m=1}^{n-1} \frac{1}{m-1+\log m+A(y_1)}. \end{aligned} \quad (24)$$

Adding and subtracting the harmonic number H_{n-1} and defining the sequence

$$S_{n-1} = \sum_{m=1}^{n-1} \frac{\log m + A(y_1) - 1}{m(m-1+\log m+A(y_1))}, \quad (25)$$

the following lower bound results

$$\begin{aligned} y_n &\geq y_1 + n - 1 + H_{n-1} - \sum_{m=1}^{n-1} \left(\frac{\log m + A(y_1) - 1}{m(m-1+\log m+A(y_1))} \right) = y_1 + n - 1 + H_{n-1} - S_n \\ &\geq n + \log n + y_1 - 1 + \gamma - \frac{1}{2n} - \frac{1}{12n^2 - S_{n-1}}. \end{aligned} \quad (26)$$

To summarize these results, it has been shown that

$$n + \log n - S_{n-1} + A(y_1) - \frac{1}{2n} - \frac{1}{12n^2} \leq y_n \leq n + \log n + A(y_1) - \frac{1}{2n}. \quad (27)$$

Since $\log n/n \rightarrow 0$ as n gets large it follows that $\lim_{n \rightarrow \infty} y_n/n = 1$. For large n we can also write

$$\log n + A(y_1) - S_{n-1} - \frac{1}{2n} \leq y_n - n \leq \log n + A(y_1) - \frac{1}{2n}.$$

Dividing through by $\log n$, this becomes

$$1 + \frac{A(y_1) - S_{n-1}}{\log n} - \frac{1}{2n \log n} \leq \frac{y_n - n}{\log n} \leq 1 + \frac{A(y_1)}{\log n} - \frac{1}{2n \log n}. \quad (28)$$

By the squeeze theorem, result (28) implies that

$$\lim_{n \rightarrow \infty} \frac{y_n - n}{\log n} = 1. \quad (29)$$

Limit (29) can be presented in terms of the sequence x_n by going back to the definition $y_n = 1/x_n$ and simplifying. It is equivalent to

$$\lim_{n \rightarrow \infty} \frac{1 - x_n n}{x_n \log n} = 1. \quad (30)$$

The rules for limits can be used to produce other limits related to (30). Since $nx_n \rightarrow 1$, (30) implies as well that

$$\lim_{n \rightarrow \infty} \frac{n(1 - x_n n)}{\log n} = 1. \quad (31)$$

A Parametric Recursion

Let $\alpha > 0$ and a_0 be real constants such that $a_0 \geq 0$. Define further sequence terms $\{a_n\}$ for $n \in \mathbb{N}$ by means of the relation,

$$a_{n+1} = a_n + \alpha - \frac{a_n}{\sqrt[k]{1+a_n^k}}. \quad (32)$$

The constant α can be characterized so that sequence $\{a_n\}$ converges. First if $\{a_n\}$ converges to a limit λ , then $\alpha = \lambda/(1 + \lambda^k)^{1/k} < 1$. Thus $0 < \alpha < 1$ is necessary for the convergence of the sequence $\{a_n\}$, and it is also sufficient. Let $0 < \alpha < 1$. Define M to be

$$M = \max\{a_0, \alpha + \frac{\alpha}{(1+\alpha^k)^{1/k}}\},$$

and mapping $P(x) : [0, M] \rightarrow \mathbb{R}$ given in terms of α and k by

$$P(x) = x + \alpha - \frac{x}{(1+x^k)^{1/k}}. \quad (33)$$

If $0 \leq x \leq \alpha/(1-\alpha^k)^{1/k}$, then $0 < \alpha \leq P(x) \leq x + \alpha \leq M$. If $\alpha/(1-\alpha^k)^{1/k} \leq x \leq M$, then $0 < \alpha \leq P(x) \leq x \leq M$. This shows that $P(x)$ is a contractive mapping so that $P([0, M]) \subset [0, M]$. Differentiating $P(x)$, we find for all $x \in [0, M]$,

$$0 \leq P'(x) \leq 1 - \frac{1}{(1+x^k)^{1+1/k}} \leq 1 - \frac{1}{(1+M^k)^{1+1/k}}. \quad (34)$$

Thus (34) implies that $P(x)$ is a contraction mapping. Given $a_0 \in [0, M]$ and recursion relation defined by $a_{n+1} = P(a_n)$, the contraction mapping theorem implies that the sequence $\{a_n\}_1^\infty$ converges to the unique fixed point $\lambda \in [0, M]$ of $P(x)$. The equation $P(\lambda) = \lambda$ can be solved to give λ in terms of α and k ,

$$\lambda = \frac{\alpha}{(1-\alpha^k)^{1/k}}. \quad (35)$$

Suppose there is convergence to λ such that $\lambda \neq a_0$. Suppose $0 < \alpha < 1$ and λ is given by (35) and not equal to a_0 . Since $P'(x) > 0$, it follows that $a_n \neq \lambda$ for all n , then by definition of derivative of $P(x)$,

$$\lim_{n \rightarrow \infty} \frac{a_{n+1} - \lambda}{a_n - \lambda} = \lim_{n \rightarrow \infty} \frac{P(a_n) - P(\lambda)}{a_n - \lambda} = P'(c) = 1 - \frac{1}{(1-\lambda^k)^{1+1/k}} = 1 - (1-\alpha^k)^{1+1/k}. \quad (36)$$

When divergence takes place, an asymptotic formula for a_n can be obtained. Consider the case $\alpha > 1$ and $k > 0$.

From (32), there follows the inequality $a_{n+1} \geq a_n + \alpha - 1$ for $n \geq 0$. Iterating this relation, it follows by induction that $a_n \geq (\alpha - 1) \cdot n$ holds for all $n \geq 0$. For $n \geq 1$ on account of

$$1 - \frac{a_n}{(1+a_n^k)^{1/k}} = 1 - \frac{a_n}{a_n(1+a_n^{-k})^{1/k}} = O\left(\frac{1}{a_n^k}\right) \quad (37)$$

the following estimate holds

$$a_n = a_{n-1} + \alpha - 1 + O((n-1)^{-k}) = (\alpha - 1) \cdot n + O\left(\frac{1}{n^{k(1-n^{-1})^k}}\right). \quad (38)$$

Hence for all $n \geq 2$, upon iteration, the a_n have approximations depending on k ,

$$a_n = \begin{cases} (\alpha - 1) \cdot n + O(n^{1-k}), & \text{if } 0 < k < 1, \\ (\alpha - 1) \cdot n + O(\log n), & k = 1 \\ (\alpha - 1) \cdot n + O(1/n^{k-1}), & k > 1. \end{cases}$$

It is worth studying the case $\alpha = 1$ and $k = 2$. It is shown that $a_n/n^{1/3}$ goes like $(3/2)^{1/3}$ as $n \rightarrow \infty$. Beginning with

$$a_{n+1} = a_n + 1 - \frac{a_n}{\sqrt{1+a_n^2}}$$

$$\begin{aligned} &= a_n - \frac{a_n - \sqrt{1+a_n^2}}{\sqrt{1+a_n^2}} \\ &= a_n - \frac{a_n^2 - 1 - a_n^2}{\sqrt{1+a_n^2}(\sqrt{1+a_n^2}+a_n)} \\ &= a_n + \frac{1}{\sqrt{1+a_n^2}(\sqrt{1+a_n^2}+a_n)}, \end{aligned}$$

the following estimates hold

$$F(a_n) < a_{n+1} < R(a_n),$$

where $F(x)$ and $R(x)$ are defined by

$$F(x) = x + \frac{1}{2+2x^2}, \quad R(x) = x + \frac{1}{2x^2}.$$

Both $F(x)$ and $R(x)$ are increasing functions for $x \geq 1$. Suppose $\epsilon > 0$ and pick $n(\epsilon)$ such that for $x \geq (3/2 - \epsilon)n(\epsilon)$,

$$\frac{1-\frac{2}{3}\epsilon}{x^{2/3}} \leq \frac{1}{1+x^{2/3}},$$

followed by an $m(\epsilon)$ such that

$$a_{m(\epsilon)} > \sqrt[3]{\left(\frac{3}{2} - \epsilon\right)n(\epsilon)}.$$

Induction on n leads to the conclusion that for $n \geq 0$,

$$a_{n+m(\epsilon)} > \sqrt[3]{\left(\frac{3}{2} - \epsilon\right)(n + n(\epsilon))}. \quad (39)$$

The case $(n+1)$ of (39) holds if

$$\sqrt[3]{x + \frac{3}{2} - \epsilon} < F(\sqrt[3]{x}).$$

However, $\sqrt[3]{b+c} - \sqrt[3]{b} < c/(3\sqrt[3]{b^2})$ for $b, c > 0$, this follows from the choice of n_ϵ , so

$$\liminf_{n \rightarrow \infty} \frac{a_n}{\sqrt[3]{n}} \geq \sqrt[3]{\frac{3}{2} - \epsilon}. \quad (40)$$

Similarly choose κ_ϵ so that $a_0 < \sqrt[3]{3/2 + \epsilon}\kappa_\epsilon$ and $(1 + (2/3)\epsilon)/(x + (3/2)\epsilon)/(x + (3/2)\epsilon)^{2/3} > x^{-2/3}$ for all $x \geq (3/2 + \epsilon)\kappa_\epsilon$.

A similar induction shows that $a_n < \sqrt[3]{\left(\frac{3}{2} + \epsilon\right)(n + \kappa(\epsilon))}$ and so it follows that

$$\limsup_{n \rightarrow \infty} \frac{a_n}{\sqrt[3]{n}} \leq \sqrt[3]{\frac{3}{2} + \epsilon}. \quad (41)$$

A Geometric Construction

The sequences studied here have a geometric motivation. A large rectangle is constructed by adding a smaller rectangle of unit area to Fig. 1. To make the construction more comprehensible, a brief description is given. In the end R_M denotes a large rectangle constructed from M unit rectangles.

Start with a unit square and construct a larger rectangle by attaching the same square on the right. The length changes but the height stays fixed. Next place a third rectangle on top so length is fixed but the height changes, then a rectangle on the right which leaves the height the same but changes the base length. At this point

Fig. (1a) results. Adding a unit rectangle to the side then the top gives Fig. 1(b,c).

The rectangle which results after M steps has area M square units. Let h_M denote the height of the rectangle and λ_M the corresponding length. Now M can be even $2n$ or odd $2n + 1$, and so we conjecture that for these two cases,

$$h_{2n} = \frac{(2n-1)!}{2^{2(n-1)}(n-1)!^2}, \quad \lambda_{2n} = \frac{2^{2n}n!^2}{(2n)!}, \quad (42)$$

and

$$h_{2n+1} = \frac{(2n+1)!}{2^{2n}n!^2}, \quad \lambda_{2n+1} = \frac{2^{2n}n!^2}{(2n)!}. \quad (43)$$

These can be established by induction since they can be verified up to some value n . By induction, it is shown that h_{2n+2} , λ_{2n+2} follow by beginning with (43). To pass from rectangle R_{2n+1} to R_{2n+2} , a unit rectangle is added on the right. Thus the height of the rectangle remains constant, but not its length. Since the height remains constant, and the area is now $2n + 2$, we have

$$h_{2n+2} = \frac{(2n+1)!}{2^{2n}n!^2}, \quad \lambda_{2n+2} \cdot h_{2n+2} = 2n + 2. \quad (44)$$

Solving for λ_{2n+2} we get

$$\lambda_{2n+2} = \frac{2^{2n}(n!)^2}{(2n+1)!} (2n + 2) = \frac{2^{2n+2}(n+1)!^2}{(2n+2)!}.$$

These equations match the ones in (42) when n is replaced by $n + 1$. To go from R_{2n+2} to R_{2n+3} , a unit rectangle must be placed on top of the collection such that the length remains the same but the height changes. Consequently, if

$$\lambda_{2n+3} = \lambda_{2n+2} = \frac{2^{2n+2} \cdot (n+1)!^2}{(2n+2)!},$$

then combined with the area constraint

$$\lambda_{2n+3} \cdot h_{2n+3} = 2n + 3.$$

This result can be solved for the height,

$$h_{2n+3} = \frac{(2n+2)!}{2^{2n+2}(n+1)!^2} (2n + 3) = \frac{(2n+3)!}{2^{2n+2}(n+1)!^2}.$$

This is the result in (43) with n replaced by $n + 1$. Looking at the formulas in (42) and (43), it is clear the following relation holds

$$\lim_{n \rightarrow \infty} \frac{\lambda_{2n}}{h_{2n}} = \lim_{n \rightarrow \infty} \frac{\lambda_{2n+1}}{h_{2n+1}}, \quad (45)$$

provided the limits exists. To prove it does and to calculate it, $\Gamma(n) = (n - 1)!$ and use Stirling's expansion (Whittaker and Watson, 2020) to the ratios in (45).

Substituting this expansion into the ratio,

$$\frac{\lambda_{2n}}{h_{2n}} = \frac{\pi}{2} \cdot \frac{e^{\theta/3n}}{e^{\theta'/12n}}.$$

Since the terms $\theta(n)$ and $\theta'(n)$ are each bounded in Stirling's formula, the limit follows immediately,

$$\lim_{n \rightarrow \infty} \frac{\lambda_{2n}}{h_{2n}} = \frac{\pi}{2}. \quad (46)$$

Using further terms in the Stirling formula, asymptotic expansions for these ratios can be calculated

$$\frac{\lambda_{2n+1}}{h_{2n+1}} = \frac{2^{4n}n!^4}{(2n-1)(2n)!} = \frac{2^{4n}\Gamma(n+1)^4}{(2n+1)\Gamma(2n+1)^2} = \frac{\pi}{2} - \frac{\pi}{8n} + \frac{5\pi}{64n^2} - \frac{11\pi}{256n^3} + O\left(\frac{1}{n^4}\right). \quad (47)$$

This expansion implies the same limit for the odd expansion as the one in (46). It is interesting to compare the coefficients of the expansion of the ratio in (46) to those of (47), we have

$$\frac{\lambda_{2n}}{h_{2n}} = \frac{2^{2n}n!^2}{(2n)!} \cdot \frac{2^{2(n-1)}(n-1)!^2}{(2n-1)!} = \frac{\pi}{2} + \frac{\pi}{8n} + \frac{\pi}{64n^2} - \frac{\pi}{256n^3} + O\left(\frac{1}{n^4}\right). \quad (48)$$

This gives the same limit as (46), and it is concluded both the even and odd ratios approach the same limit as n increases.

The collection of examples which follow are of a more general nature without reference to a specific recursion relation or case.

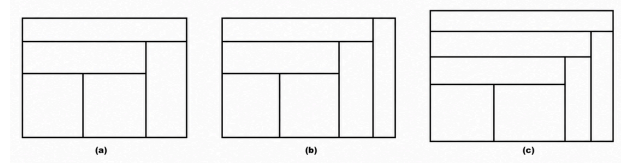


Fig. 1: Construction of rectangles R_M that correspond to equations (42) and (43).

Theorem 1

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Let n be a positive integer and define the sequence of sums $s_n = \sum_{j=1}^n a_j$. For $n \geq 2$ define S_n to be the sum

$$S_n = \sum_{k=1}^n \frac{a_k}{s_n - a_k}. \quad (49)$$

Then provided the limit $L = \lim_{n \rightarrow \infty} S_n$ exists $L \geq 1$.

Proof

Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $0 \leq u_j < 1$ and define $U_n = \sum_{k=1}^n u_k$. The following inequality holds

$$\sum_{k=1}^n \frac{u_k}{1-u_k} \geq \frac{nU_n}{n-U_n}. \quad (50)$$

Equality holds in (50) when all the u_j are equal. To see this define $v_i = 1 - u_i$, $V_n = \sum_{k=1}^n v_k$ and P_{ij} such that

$$P_{ij} = \frac{v_i}{v_j} + \frac{v_j}{v_i} = P_{ji}.$$

Since $v_i^2 - 2v_iv_j + v_j^2 = (v_i - v_j)^2 \geq 0$ with equality when $v_i = v_j$. It follows that

$$P_{ij} = \frac{v_i^2 + v_j^2}{v_i v_j} = 2 + \frac{(v_i - v_j)^2}{v_i v_j} \geq 2.$$

Sum V_n/v_i from $i = 1$ to n , we have

$$\sum_{i=1}^n \frac{V_n}{v_i} = \sum_{i=1}^n \sum_{k=1}^n \frac{v_k}{v_i} = \frac{1}{2} \cdot \sum_{i=1}^n \sum_{k=1}^n \left(\frac{v_k}{v_i} + \frac{v_i}{v_k} \right) \geq \sum_{i=1}^n \sum_{k=1}^n 1 = n^2. \quad (51)$$

Subtract nV_n from both sides of (51). It is found that

$$\sum_{j=1}^n \left(\frac{V_n}{v_j} - V_n \right) \geq n^2 - nV_n.$$

Divide both sides by V_n to get

$$\sum_{i=1}^n \left(\frac{1}{v_i} - 1 \right) \geq \frac{n(n-V_n)}{V_n}.$$

Since $V_n = \sum_{i=1}^n v_i = \sum_{i=1}^n (1 - u_i) = n - U_n$, this is equivalent to

$$\sum_{i=1}^n \left(\frac{u_i}{1-u_i} \right) \geq \frac{nU_n}{n-U_n}. \quad (52)$$

Let $\{a_n\}_{n=1}^\infty$ be a sequence of positive real numbers such that $s_n = \sum_{k=1}^n a_k$. Define the sequence $\{u_k\}_{k=1}^n$ in terms of the a_k to be

$$u_k = \frac{a_k}{s_n}.$$

The sequence $\{u_k\}_{k=1}^n$ defined this way satisfies $0 \leq u_k < 1$ such that $U_n = \sum_{k=1}^n u_k = 1$. It is concluded from (52) that the following holds for any $n \geq 2$,

$$S_n = \sum_{i=1}^n \left(\frac{a_i}{s_n - a_i} \right) = \sum_{i=1}^n \left(\frac{a_i/s_n}{1 - a_i/s_n} \right) = \sum_{i=1}^n \frac{u_i}{1 - u_i} \geq \frac{n}{n-1}.$$

Provided $\lim_{n \rightarrow \infty} S_n = L$ exists, this result implies that L satisfies the lower bound

$$L = \lim_{n \rightarrow \infty} S_n \geq \lim_{n \rightarrow \infty} \frac{n}{n-1} = 1.$$

Theorem 2

Let $\{a_n\}_{n=1}^\infty$ be a strictly increasing sequence of real numbers which satisfy a constraint $a_n \leq n^2 \cdot \ln(n)$ for all $n \leq 1$. Then the infinite series

$$S = \sum_{n=1}^\infty \frac{1}{a_{n+1} - a_n} \quad (53)$$

diverges.

Proof

The sequence $\{a_n\}_{n=1}^\infty$ is strictly increasing by assumption. This means $a_n < a_{n+1}$ and the a_n satisfy the condition $a_n \leq \ln(n)$. Consider the subset of those sequences $\{\alpha_n\}_{n=1}^\infty$ of $\{a_n\}_{n=1}^\infty$ which satisfy the inequalities

$$\alpha_{n-1} \leq (n-1)^2 \cdot \ln(n-1) \leq \alpha_n \leq n^2 \ln(n) < a_{n+1}. \quad (54)$$

The subset $\{\alpha_n\}$ of these sequences give rise to the largest denominators in S . If series composed of these sequences fail to converge, then none of the others

in $\{a_n\}$ converge since $1/(a_{n+1} - a_n) \geq 1/(\alpha_{n+1} - \alpha_n)$. Consider series of the form

$$\sum_{n=1}^\infty \frac{1}{\alpha_{n+1} - \alpha_n}. \quad (55)$$

Using the bounds in (54), the partial sums of (54) satisfy the inequality

$$\sum_{n=1}^N \frac{1}{\alpha_{n+1} - \alpha_n} \geq \sum_{n=1}^N \frac{1}{(n+1)^2 \ln(n+1) - n^2 \ln(n)}.$$

The function $h = x^2 \ln(x)$ is differentiable on $[1, \infty)$, so for each $n \geq 1$, the mean value theorem implies there exists $c_n \in (n, n+1)$ such that

$$\frac{(n+1)^2 \ln(n+1) - n^2 \ln(n)}{n+1 - n} = c_n(2 \ln(c_n) + 1) < (n+1)(2 \ln(n+1) + 1).$$

It follows that

$$\sum_{n=1}^N \frac{1}{\alpha_{n+1} - \alpha_n} > \sum_{n=1}^N \frac{1}{(n+1)(2 \ln(n+1) + 1)}.$$

The convergence of the series on the right can be established by means of the integral test. By means of the substitution $u = \ln(x+1) + 1/2$, it follows that

$$\int_1^N \frac{dx}{(x+1)(\ln(x+1)+1/2)} = \ln(\ln(N+1) + \frac{1}{2}) - \ln(\ln(2) + \frac{1}{2}).$$

Since $\ln(\ln(N+1)) \rightarrow \infty$ as $N \rightarrow \infty$, it follows that the integral and hence the related series (55) diverge together as $N \rightarrow \infty$.

Theorem 3

Let β be a real number such that $\beta > 1$ and $\{u_n\}$ is a sequence of positive numbers which satisfy $\lim_{n \rightarrow \infty} u_n = 0$, also $\lim_{n \rightarrow \infty} (u_n - u_{n+1})/u_n^\beta$ exists and is nonzero. Then the series $\sum_{n=1}^\infty u_n$ converges if and only if $\beta < 2$.

Proof

Define the sequence $\sigma_n(\beta)$ as

$$\sigma_n(\beta) = \frac{u_n - u_{n+1}}{u_n^\beta}. \quad (56)$$

By assumption $\lim_{n \rightarrow \infty} \sigma$ exists and $\sigma \neq 0$. Let S be the set of all series consisting of positive terms tending to zero and both the sequence σ_n and series $\sum_{n=1}^\infty u_n$ converge.

Suppose the series $\sum_{n=1}^\infty u_n$ converges and hence an element of S . A class of series in S are p -series whose terms are defined by $u_n = 1/n^p$. All terms are positive with $\lim_{n \rightarrow \infty} u_n = 0$ for $p > 0$ and $\sum_{n=1}^\infty n^{-p}$ converges only for $p > 1$. To ensure p -series belong to S , the convergence of sequence σ_n is studied. Substitute $u_n = n^{-p}$ into (56)

$$\sigma_n(\beta) = n^{p\beta} \left(\frac{1}{n^p} - \frac{1}{(n+1)^p} \right) = n^{p(\beta-1)-1} \left(p - \frac{p(p+1)}{2n} + O\left(\frac{1}{n^2}\right) \right).$$

Sequence σ_n converges to a specific non-zero limit only for a specific value of β which given in terms of p is $\beta = 1 + \frac{1}{p}$.

As $p \rightarrow 1^+$, this implies that $\beta \rightarrow 2^-$. As soon as $\alpha \geq 2$, the sequence σ_n diverges for all $p > 1$. Therefore for p -series to be in S , $\beta \in (1, 2)$.

Suppose $1 < \beta < 2$. It is required to show the series $\sum_{n=1}^{\infty} \sigma_n$ converges, which means it is in S . Since $u_n \rightarrow 0^+$, it must be that $u_{n+1} < u_n$ infinitely often so $\sigma_n > 0$. Consequently $\sigma^+ = \limsup \sigma_n > 0$.

However, $\lim_{n \rightarrow \infty} \sigma_n = 0$ exists so $\liminf \sigma_n$ must agree with σ^+ which gives $\sigma > 0$. Consequently, there exists an M such that $\sigma_n > 0$ for all $n > M$. For $\alpha - 1 \in (0, 1)$, by the least upper bound property, choose $N > n > N^{\beta-1} > M$ such that

$$u_n \sigma_n^{\frac{1}{\beta-1}} > \frac{1}{N}.$$

Then the inequalities

$$n u_n^{\beta-1} \sigma_n > \frac{n}{N^{\beta-1}} > 1$$

follows. In fact, suppose we set $N = n + 1$, then this inequality becomes

$$n u_n^{\beta-1} \sigma_n > \frac{n}{(n+1)^{\beta-1}} = \frac{n^{2-\beta}}{(1+1/n)^{\beta-1}} > 1.$$

From (56), $\sigma_n^\beta, \sigma_n^\beta = u_n - u_{n+1}$. Hence as $n \rightarrow \infty$, it follows that

$$n \left(\frac{u_{n+1}}{u_n} - 1 \right) = -n u_n^{\beta-1} \sigma_n < -1.$$

Theorem 4

Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers and define the partial sums $s_n = \sum_{k=1}^n z_k$. and suppose all s_n are nonzero.

(i) Given that s_n does not approach zero as n increases, show that the series diverges:

$$\sum_{n=1}^{\infty} \frac{z_n}{s_n} \quad (57)$$

(ii) Show that if s_n tends to a limit and $s - s_n$ never vanishes, then the series diverges:

$$\sum_{n=1}^{\infty} \frac{z_n}{s - s_{n-1}} \quad (58)$$

Proof

(i) Since $s_n = z_1 + z_2 + \dots + z_n$ we have $s_n - s_{n-1} = z_n$. Therefore, for any k ,

$$1 - \frac{z_k}{s_k} = \frac{s_{k-1}}{s_k}.$$

It suffices to suppose the sequence $\{z_n\}_1^{\infty}$ starts with a nonzero z_1 . Taking the product on both sides with respect to $k = 2, \dots, n$, the product on the right telescopes and an explicit expression for the partial product P_n is obtained

$$P_n = \left(1 - \frac{z_2}{s_2}\right) \cdot \left(1 - \frac{z_3}{s_3}\right) \cdots \left(1 - \frac{z_n}{s_n}\right) = \frac{s_1}{s_2} \cdot \frac{s_2}{s_3} \cdots \frac{s_{n-1}}{s_n} = \frac{s_1}{s_n} = \frac{z_1}{s_n}.$$

Assuming s_n does not tend to zero as $n \rightarrow \infty$, convergence of the infinite product $P = \lim_{n \rightarrow \infty} P_n$ enquires that the series $\sum_1^{\infty} \frac{z_n}{s_n}$ as well as the limit $\lim_{n \rightarrow \infty} s_n = \sum_1^{\infty} z_k$ exists, that is, the series converges. Otherwise they diverge together.

(ii) Define the sequence $u_n = \sum_{k=n}^{\infty} z_k$ such that $u_1 = s$. Then we can write

$$1 - \frac{z_k}{u_k} = \frac{u_k - z_k}{u_k} = \frac{u_{k+1}}{u_k}.$$

Calculate the partial product of this starting from $k = 1, \dots, n$ with s finite

$$\Pi_n = \left(1 - \frac{z_1}{u_1}\right) \left(1 - \frac{z_2}{u_2}\right) \cdots \left(1 - \frac{z_n}{u_n}\right) = \frac{u_{n+1}}{u_1} = \frac{u_{n+1}}{s}.$$

The right-hand side tends to zero since $u_{k+1} \rightarrow 0$ when s is finite. Thus the infinite product approaches zero as $n \rightarrow \infty$. If a divergent product has value zero, it is said that it diverges to zero. This behavior of the infinite product implies that the series

$$\sum_{n=1}^{\infty} \frac{z_n}{u_n} = \sum_{n=1}^{\infty} \frac{z_n}{\sum_{k=n}^{\infty} z_k}$$

must diverge as well.

Theorem 5

Let $x_1, x_2, \dots$ be a sequence of positive numbers such that

$$\lim_{n \rightarrow \infty} \frac{\log(x_n)}{x_1 + \dots + x_n} \quad (59)$$

is a negative number. Then the limit

$$\lim_{n \rightarrow \infty} \frac{\log(x_n)}{\log(n)} = -1. \quad (60)$$

Proof

Since (59) is negative and the denominator is positive, the logarithm becomes negative at some point $n > N_0$. If the denominator goes to ∞ as $x_n \rightarrow 0$ the ratio is indeterminate. If the denominator converges, the x_n should converge so in both cases, the limit is defined and negative. Suppose the limit exists and satisfies $-\gamma < 0$ with $\gamma > 0$ as well. It is shown that $\gamma = 1$ follows. Let n be even, or if n is odd, replace n by $n - 1$ and arrive at

$$\frac{\log(x_n) - \log(x_{n-1}) + \dots + \log(x_1) - \log(x_1)}{\log(n) - \log(n-1) + \dots + \log(2) - \log(1)} = \frac{\log(x_n)}{\log(n)} - \frac{\log(x_1)}{\log(n)}. \quad (61)$$

The limit on the right of (61) is $-\alpha$ under the assumption since x_1 is fixed. Hence the limit on the left is $-\gamma$. Setting $a_n = \log(x_n) - \log(x_{n-1})$ and $b_n = \log(n) - \log(n-1) > 0$. The Cesaro-Stoltz Lemma (Polya and Szego, 1972) lets us conclude that

$$\lim_{n \rightarrow \infty} \frac{\log(x_n) - \log(x_{n-1})}{\log(n) - \log(n-1)} = -\gamma. \quad (62)$$

Now (59) satisfies

$$\lim_{n \rightarrow \infty} \frac{\log(x_n)}{x_1 + x_2 + \dots + x_n} = -\alpha.$$

where $\alpha > 0$ is finite. Given any $\epsilon > 0$, there exists an M such that for all $m > M + 1$

$$-\alpha x_m = 2\epsilon \sum_{k=1}^{m-1} x_k - \gamma x_m < \log(x_m) - \log(x_{m-1}) < -\alpha x_m + 2\epsilon \sum_{k=1}^{m-1} x_k + \epsilon x_m. \quad (63)$$

For m sufficiently large, Taylor's theorem applied to $\log(1-x)$ at $x = 1/m$ allows us to state

$$\log(m) - \log(m-1) = -\log(1 - \frac{1}{m}) = \frac{g_m}{m},$$

such that $g_m \rightarrow 1$ as $m \rightarrow \infty$. Using (62) there must be M_0 such that for any $\epsilon > 0$ and $m > M + 1$

$$(-\gamma - \epsilon) \frac{g_m}{m} < \log(x_m) - \log(x_{m-1}) < (-\gamma + \epsilon) \frac{g_m}{m}. \quad (64)$$

Take $N = \max(M, M_0)$ and require that $m > N$ in the following. Combining (64) and (66) from one direction of the inequalities implies that

$$-\alpha x_m - 2\epsilon \sum_{k=1}^{m-1} x_k - \gamma x_m < (-\gamma + \epsilon) \frac{g_m}{m}.$$

Rearranging this and dividing by α produces a lower bound for x_m

$$x_m > (\gamma - \epsilon) \frac{g_m}{\alpha m} - \frac{2}{\alpha} \epsilon \sum_{k=1}^{m-1} x_k - \frac{\gamma}{\alpha} x_m.$$

Similarly an upper bound for x_m is found

$$x_m < (\gamma + \epsilon) \frac{g_m}{\alpha m} + \frac{2}{\alpha} \epsilon \sum_{k=1}^{m-1} x_k + \frac{1}{\alpha} \epsilon x_m.$$

Summarizing these results, x_m satisfies the bounds

$$\frac{1}{\alpha} (\gamma - \epsilon) \frac{g_m}{m} - \frac{2}{\alpha} \epsilon \sum_{k=1}^{m-1} x_k - \epsilon \frac{x_m}{\alpha m} < x_m < \frac{1}{\alpha} (\gamma + \epsilon) \frac{g_m}{m} + \frac{2}{\alpha} \epsilon \sum_{k=1}^{m-1} x_k + \epsilon \frac{x_m}{\alpha m}. \quad (65)$$

Define the sequence $s_m = (x_1 + \dots + x_{m-1})/m$. It is known that $s_m + (x_m/m) \rightarrow 0$ when $x_m \rightarrow 0$ (Polya and Szego, 1972). Hence $s_m \rightarrow 0$ and as well $(\gamma/\alpha) g_m \rightarrow (\gamma/\alpha) > 0$. Take N sufficiently large so $\epsilon > 0$ can be chosen to be $\epsilon = 1/m^2$ which will suffice. From (64)

$$\frac{1}{\alpha} (\gamma - \frac{1}{m^2}) \frac{g_m}{m} < x_m + \frac{2}{\alpha m} s_m + \frac{x_m}{\alpha m^2}. \quad (66)$$

Take the logarithm of both sides to get

$$\log(\frac{\gamma}{\alpha} - \frac{1}{\alpha m^2}) + \log g_m - \log m < \log(x_m + \frac{2}{\alpha m} s_m + \frac{x_m}{\alpha m^2}).$$

Therefore

$$\begin{aligned} -1 &= \lim_{m \rightarrow \infty} [\log(\frac{\gamma}{\alpha} - \frac{1}{\alpha m^2}) + \log(g_m)] \frac{1}{\log(m)} - \frac{\log(m)}{\log(m)} \leq \lim_{m \rightarrow \infty} [\log(x_m + \frac{2}{\alpha m} s_m + \frac{x_m}{\alpha m^2})] \frac{1}{\log(m)} \\ &= \lim_{m \rightarrow \infty} \frac{\log(x_m)}{\log(m)}. \end{aligned} \quad (67)$$

The last equality follows since the last two terms in the logarithm are down by a factor of $1/m$ from the first x_m . Similarly, on the other side

$$x_m - \frac{2}{\alpha m} s_m - \frac{x_m}{\alpha m^2} < (\frac{\gamma}{\alpha} - \frac{1}{\alpha m^2}) \frac{g_m}{m}.$$

Taking the limit,

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{\log(x_m)}{\log(m)} &= \lim_{m \rightarrow \infty} [\log(x_m - \frac{2}{\alpha m} s_m - \frac{x_m}{\alpha m^2})] \frac{1}{\log(m)} \leq \lim_{m \rightarrow \infty} [(\log(\frac{\gamma}{\alpha} - \frac{1}{\alpha m^2}) \\ &+ \log(g_m)) \frac{1}{\log(m)} - \frac{\log(m)}{\log(m)}] = -1. \end{aligned} \quad (68)$$

Combining (67) and (68), the squeeze theorem gives result (60).

Theorem 6

Sequences and series share common features, for example, the partial sums of a series generate a kind of sequence. This last theorem makes use of Abel's lemma, a very useful lemma in general that can be used to prove Dirichlet's test for convergence.

Suppose that $a_0 > a_1 > a_2 > \dots$ and define the sequence S_n of partial sums

$$S_n = \sum_{k=n}^{\infty} (-1)^{k-n} a_k = a_n - a_{n+1} + a_{n+2} - \dots. \quad (69)$$

Then $\sum_{n=0}^{\infty} a_n S_n < \infty$ if and only if $\sum_{n=0}^{\infty} a_n^2 < \infty$.

Proof

Since $\{a_n\}$ are monotonically decreasing to zero, the S_n are finite and well defined by the alternating series test. The partial sums $S_{n,r} = \sum_{j=n}^r (-1)^{n-j} a_j$ are positive for all $r \geq n$. It follows the S_n are positive for $n \in \{0\} \cup \mathbb{N}$. The identities are useful with $S = \sum_{k=0}^{\infty} a_k$, $A_n = \sum_{k=0}^n (-1)^k a_k$ and S_n is given in (69)

$$(-1)^n S = \sum_{j=0}^{\infty} (-1)^{j-n} a_j = \sum_{j=0}^n (-1)^{j-n} a_j + \sum_{j=n+1}^{\infty} a_j = (-1)^n A_n - S_{n+1}, \quad (70)$$

$$(-1)^n A_n = (-1)^n A_{n-1} + a_n = (-1)^n S - S_n + a_n. \quad (71)$$

Abel's lemma is required: Suppose $(a_n)_0^{\infty}$ and $(b_n)_0^{\infty}$ are sequences in $\mathbb{C}$ and define $P_n = \sum_{k=0}^n a_k$, then

$$\sum_{n=p+1}^q a_n b_n = \sum_{n=p+1}^q P_n (b_n - b_{n+1}) + P_q b_{q+1} - P_p b_{p+1}.$$

In the lemma, substitute $a_n = b_n = (-1)^n a_n$, so the product is a_n^2 and

$$\begin{aligned} \sum_{n=p+1}^q a_n^2 &= \sum_{n=p+1}^q A_n((-1)^n - (-1)^{n+1} a_{n+1}) + \\ &A_q(-1)^{q+1} a_{q+1} - A_p(-1)^{p+1} a_{p+1} \\ &= \sum_{n=p+1}^q (-1)^n A_n(a_n + a_{n+1}) + (-1)^{q+1} A_q a_{q+1} + \\ &(-1)^p A_p a_{p+1}. \end{aligned} \quad (72)$$

The sums are finite and A_n can be eliminated in terms of S_{n+1} . Using (70) in (72), the following bound results since $a_n > a_{n+1}$ and $S_n > 0$

$$\begin{aligned} \sum_{n=p+1}^q a_n^2 &= \sum_{n=p+1}^q (-1)^n (a_n + a_{n+1}) + \\ &(-1)^{q+1} A_q a_{q+1} + (-1)^p A_p a_{p+1} \\ &\geq ((-1)^{p+1} a_{p+1} + (-1)^q a_{q+1}) S + \\ &2 \sum_{n=p+1}^q S_{n+1} a_{n+1} + (-1)^{q+1} A_q a_{q+1} + (-1)^p A_p a_{p+1}. \end{aligned}$$

Since $c a_n \rightarrow 0$ for a constant c and A_q converges to S as $q \rightarrow \infty$, the first, third and fourth terms on the right of the inequality remain bounded as q increases. The assumption $\sum_{n=0}^{\infty} a_n^2 < \infty$ implies that $\sum_{n=p+1}^{\infty} a_n^2 < \infty$, so the series

$$\sum_{m=p+2}^{\infty} S_m a_m < \infty.$$

This convergence implies that $\sum_{m=0}^{\infty} S_m a_m < \infty$.

From (71) using (72), one obtains

$$\begin{aligned} \sum_{n=p+1}^q a_n^2 &= \sum_{n=p+1}^q (-1)^n (a_n + a_{n+1}) S - \\ \sum_{n=p+1}^q S_n (a_n + a_{n+1}) &+ \sum_{n=p+1}^q a_n (a_n + a_{n+1}) \\ &+ (-1)^{q+1} A_q a_{q+1} + (-1)^{p+1} A_p a_{p+1} \end{aligned}$$

The third term in this is moved to the left side and multiplying by -1 gives

$$\begin{aligned} \sum_{n=p+1}^q a_n a_{n+1} &= ((-1)^p a_{p+1} + (-1)^{q+1} a_{q+1}) + \\ \sum_{n=p+1}^q S_n (a_n + a_{n+1}) &+ (-1)^q A_q a_{q+1} + \\ &(-1)^{p+1} A_p a_{p+1} \\ &\leq ((-1)^p a_{p+1} + (-1)^{q+1} a_{q+1}) S + 2 \sum_{n=p+1}^q S_n a_n + \\ &(-1)^q A_q a_{q+1} + (-1)^{p+1} A_p a_{p+1}. \end{aligned}$$

Suppose that $\sum_{n=p+1}^q a_n S_n < \infty$ converges, since the limits of all other terms on the right exist, as $q \rightarrow \infty$,

$$\sum_{n=p+1}^{\infty} a_n a_{n+1} < \infty. \quad (73)$$

The a_n are all positive and decrease monotonically, so series (73) is an upper bound for the sum of interest

$$\sum_{n=p+2}^{\infty} a_n^2 = \sum_{n=p+1}^{\infty} a_{n+1}^2 < \sum_{n=p+1}^{\infty} a_n a_{n+1} < \infty. \quad (74)$$

This implies that $\sum_{n=0}^{\infty} a_n^2 < \infty$, which is the second implication of the theorem.

Conclusions

It has been seen that many recursively defined sequences can be studied in closed form and if there is a limit, it can often be evaluated. Theorems present a synthesis of ideas that often follow from examples and encompass large classes of cases. Many of these ideas and techniques can be extended and applied to the study of convergence of infinite series.

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